Time-consistent pension policy with minimum guarantee and sustainability constraint*

Hillairet Caroline, † Kaakaï Sarah ‡ Mrad Mohamed §

Abstract

This paper proposes and investigates an optimal pair investment/pension policy for a pay-as-you-go (PAYG) pension scheme. The social planner can invest in a buffer fund in order to guarantee a minimal pension amount. The model aims at taking into account complex dynamic phenomena such as the demographic risk and its evolution over time, the time and age dependence of agents preferences, and financial risks. The preference criterion of the social planner is modeled by a consistent dynamic utility defined on a stochastic domain, which incorporates the heterogeneity of overlapping generations and its evolution over time. The preference criterion and the optimization problem also incorporate sustainability, adequacy and fairness constraints. The paper designs and solves the social planner's dynamic decision criterion, and computes the optimal investment/pension policy in a general framework. A detailed analysis for the case of dynamic power utilities is provided.

Keywords: Consistent Dynamic Utility, PAYG Pension Policy, Sustainability and Actuarial Fairness, Demographic and Financial risk sharing, Stochastic Control.

Introduction

Pay-as-you-go (PAYG) systems in aging countries face serious challenges caused by both the decrease in birth rates and an unprecedented increase in the life expectancy (see e.g. [ABH02], [CDRV09], [Mas12]). Intergenerational solidarity is one of the main pillars of pay-as-you-go (PAYG) pension plans, in which contributions paid by working participants are redistributed to current pensioners, inducing risk sharing between generations. However, the

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†CREST, UMR CNRS 9194, ENSAE, Institut Polytechnique de Paris
‡Laboratoire Manceau de Mathématiques, Institut du Risque et de l'Assurance, Le Mans Université
§LAGA, UMR CNRS 7539, Université Sorbonne Paris Nord

sustainability of PAYG systems has become a key challenge for policymakers in aging populations. On the other hand, pension systems should provide adequate benefits for retirees as well as an acceptable level of fairness between generations, as underlined in [AGDCBPD18]. Nevertheless, these features are not necessarily compatible in a genuine PAYG system which imposes strict budget constraints.

Our goal is to propose an adaptive decision criterion in order to design an optimal policy that is consistent with both sustainability and adequacy constraints. We consider a PAYG system with defined contributions, in which the social planner has the flexibility to invest to/borrow from a buffer fund and pensioners are guaranteed a minimum annuitized pension amount. The buffer fund allows the social planner to invest in the financial market and mitigate the demographic risk. An important challenge is to convey the complexity of the problem, by taking into account key phenomena such as the demographic risk and its evolution over time, the time and age dependence of agents' preferences, or financial risks. To the best of our knowledge, these problems have only been tackled either partially or separately in the literature.

We adopt a dynamic and continuous time approach, which incorporates the heterogeneity of overlapping generations and the non stationary evolution of the population over time. When annuitization of retirement is not considered (that is at retirement, individuals receive a lumpsum, see e.g. [Gol08], [GG12]), retirees bear their own longevity risk. In this paper, the uncertain future increase in the duration of life is taken into account by the social planner, who redistribute the longevity risk across generations. [DMMS06] consider a population with both fixed dependency ratio and retirement period. In [CDJP11], a model with 55 overlapping generations is considered but individuals all die at the age of 80, and the population age composition is stationary. [Boh01] and [DM15] consider models with stochastic demographic shocks, but in two-period models. [AGD19] use a similar McKendrick-Von Foerster flexible population dynamics model as in this paper, however with deterministic age and time dependent birth and mortality rates, while we consider stochastic demographic rates in our setting. This allows us to take into account stylized fact of the population dynamics, such as uncertain longevity or dependency ratio increases.

Part of the literature on PAYG pension design take on an macro, general-equilibrium viewpoint (see e.g. [Boh01], [Boh06]). Alternatively to this general-equilibrium approach, another trend in the literature considers an approach where demographic and wages processes as well as the financial market are exogeneous. For instance, [GODCBPH16] and [AGD19] (see also [HZ02]) study deterministic models of PAYG pension systems, in which the social planner can invest in a buffer fund with known returns. The optimal pension policy is derived from optimizing solvency indicators, with no adequacy constraints. [GG12] study the social planner's optimal pension policy and investment strategy in a complete market, in a stochastic model where the pension benefit is a lumpsum at retirement with a minimum guaranteed, and with a sustainability constraint on the buffer fund at a given terminal time. Nevertheless, it is obviously impossible to hedge perfectly the demographic and economic risks through the financial market, that is why we consider in this paper an incomplete financial market in which the social planner may invest/borrow. Besides, the sustainability of the pension scheme is ensured by imposing a pathwise solvency constraint on the buffer fund (see also [DGG11] for a similar assumption in the case of a fully funded pension system).

As the representative of past, present and future generations, the social planner should aggregate preferences of all pensioners. This aggregation is the key in the fairness criterion as this benevolent social planner aims at dealing with successive overlapping generations fairly. Thus, the social planner's decision criterion that appears in the optimization problem's formulation is composed of the buffer fund utility and an aggregated utility which should capture the heterogeneous preferences of different generations, thus is bound to be complex ([EKHM17]).

The formulation of problem is related to the literature on optimal investment and consumption with labor income, respectively corresponding in our setting to the buffer fund, pensions and contributions. The literature usually states this optimization problem in a backward formulation, see e.g. [HP93], [Cuo97], [EKJP98], or more recently [MS20] for a general setting in incomplete markets. The backward approach has several drawbacks when considering the framework of PAYG pensions. First it does not incorporate any changes in the agents' preferences, or any uncertain evolution of the environment variables. Furthermore, in our context of intergenerational risk-sharing for pensions, fixing a time-horizon is difficult and can lead to optimal choices that depend on the time horizon, inducing artificial phenomena such as the liquidating the fund at terminal time (see e.g. [GG12]). Finally, the social planner should be able to identify his/her preference at any intermediate time, in order to ensure consistency across time and generations. Introduced by Musiela and Zariphopoulou [MZ08, MZ11], the framework of dynamic utilities is adapted to solve the issues raised above, by proposing long-term, time-coherent policies. Dynamic utilities allow us to define adaptive strategies adjusted to the information flow, in non-stationary and uncertain environment. A framework for forward dynamic utilities of investment and consumption have been introduced in [EKHM18]. The present paper is built upon [EKHM18] in order to obtain new results in the presence of stochastic pathwise constraints on the buffer fund (wealth) and pensions (consumption).

The population model, PAYG pension system with sustainability and adequacy constraints, and incomplete financial market are introduced in Section 1. The social planner's dynamic decision criterion is formulated in Section 2, by introducing dynamic utilities defined on stochastic domains. In particular, we give sufficient conditions on the local characteristics of the dynamic utility for the utility to be well defined. The aggregation of pensioners' preferences is introduced in Section 2.2. The main results of the paper are presented in Section 3. We first introduce a natural consistency HJB-SPDE derived from the dynamic programming principle, so that the decision criterion is consistent over time. Under this sufficient condition, the optimal constrained investment and pension policy is derived explicitly in Theorem 3.5. The remainder of the section is dedicated to proving this result. Section 4 details two examples in the case of dynamic power utilities.

1 The model

All stochastic processes are defined on a standard filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the natural filtration generated a n-dimensional Brownian motion W, and that is assumed to be right continuous and complete.

1.1 Population dynamics

In order to include the stylized facts and non-stationarity of an aging population evolution, we consider a population structured continuously in age and time. At time t, the number of individuals of age a is denoted by $n(t, a)$, and the demographic (birth and mortality) rates are modeled by a family of $\mathbb{F}$-adapted nonnegative processes $(d(\cdot, a))_{a\geq 0}$ and $(b(\cdot, a))_{a\geq 0}$, uniformly bounded and continuous in age.

Between a small period of time $[t, t + dt]$, a (random) proportion $d(t, a)dt$ of individuals of age a die, while individuals of age a give birth to $b(t, a)n(t, a)$ individuals. Formally, the population dynamics is described by a partial differential equation with stochastic coefficients, generalizing the standard McKendrick-Von Foerster equations:

$$(\partial_t + \partial_a)n(t, a) = -d(t, a)n(t, a), \tag{1.1}$$

$$n(0, t) = \int_0^\infty b(t, a)n(t, a)da. \tag{1.2}$$

A detailed analysis of such equations can be found e.g. in [Hop75], [Web85] or [Bou16]. In particular, if the initial number of individuals $N_0 = \int_0^\infty n(0, a)da$ is finite, then the population stays finite over time.

This model allows for a more realistic description of the population than in standard discrete time overlapping generation models. Time-dependent stochastic birth and mortality rates can describe the uncertain aging of the population, by taking into account phenomena such as the decrease of mortality rates over time or birth rates declines. In addition, the modeling can be extended to include intra-cohort heterogeneity or exogenous population flows. Finally, we consider in this paper a single-sex model only for ease of notation.

Workers are assumed to enter the work force at fixed age a_e, and retires at age a_r. Thus, the number of pensioners at time t is

$$N_t^r = \int_{a_r}^\infty n(t, a)da, \tag{1.3}$$

social and the number of workers is

$$N_t^w = \int_{a_e}^{a_r} n(t,a)da. \tag{1.4}$$

In the following, all results can be straightforwardly extended when the retirement age $a_r(t)$ (or $a_e(t)$) depends on time. For ease of notations, the time variable is omitted in the following.

1.2 Pension system

We define below the cumulative contribution rate process C_t payed by the workers at time t, and the cumulative pension rate process P_t received by the pensioners at time t. In what follows, we assume that individuals in a given cohort (of age a at time t) receive the same wage and pension amount. In the case of heterogeneity inside the cohort, the age-dependent wage and pension processes can be interpreted as the average amounts over the cohort.

Contribution process The wage process of any worker of age $a \in [a_e, a_r]$ at time t is assumed to be an exogenous $\mathbb{F}$-adapted process $(\mathfrak{e}_t(a))$. All workers contribute a fixed proportion $\alpha_c \in [0,1]$ of their wage in order to finance pensions of current pensioners. The cumulative contribution process (C_t) is thus defined by

$$C_t = \alpha_c \int_{a_e}^{a_r} \mathfrak{e}_t(a) n(t,a)da, \quad \forall\, t \geq 0. \tag{1.5}$$

Il the wage does not depend on the age, then $C_t = \alpha_c \mathfrak{e}_t N_t^w$.

Pension process and adequacy constraint In a strict PAYG system, the cumulative pension amount P_t to be paid at time t is only financed by the current contributions C_t, inducing the budget constraint:

$$C_t = P_t, \quad \forall t \geq 0.$$

In the case of defined contributions that we consider here, the worker's contributions are fixed, and thus pensioners bear all the economic (wages) and demographic risk. For instance, the retirement income can be drastically reduced following a decrease in the working population or an increase of the retired population. In order to achieve a minimum adequacy of retirement incomes, we consider the following pension mechanism:
Each pensioner of age a at time t receives a pension amount $p_t(a)$ depending on her age, with

$$p_t(a) = p_t^{min}(a)\rho_t, \quad a \geq a_r,\ t \geq 0. \tag{1.6}$$

$p_t^{min}(a)$ corresponds to a minimum pension amount guaranteed to pensioners (see examples below) and (ρ_t) represents a global adjustment with respect to the benchmark, which is the same for all pensioners living at time t. The total pension amount is thus

$$P_t = \rho_t P_t^{min}, \text{ with } P_t^{min} = \int_{a_r}^{\infty} p_t^{min}(a) n(t,a) da. \tag{1.7}$$

The pension process should satisfy the **adequacy constraint** (1.8) below:

$$\forall t \geq 0, \forall a \geq a_r,\ p_t(a) \geq p_t^{min}(a) \text{ a.s., } \quad \text{or equivalently } \rho_t \geq 1 \text{ a.s.} \tag{1.8}$$

In the following, two examples are studied as an application of the general results:

Example 1. When $p_t^{min}(a) \equiv p_t^{min}$, all pensioners at time t receive the same pension amount $p_t^{min}\rho_t$ and the minimal cumulative pension amount is $P_t^{min} = N_t^r p_t^{min}$. For example, p_t^{min} can be indexed on current wages, contributions, or any other indicator.

Example 2. A more realistic choice for p^{min} is to take a base pension computed at retirement date, multiplied by an indexation factor:

$$p_t^{min}(a) = p_{ret}(a_r + t - a)\ e^{\int_{a_r+t-a}^{t} \lambda_s ds}. \tag{1.9}$$

• $p_{ret}(s)$ corresponds to the base pension amount received by an individual retiring (i.e. of age a_r) at time s. Then $p_{ret}(a_r + t - a)$ is the pension amount received at retirement time by an individual of age a at time t. For instance, $p_{ret}(s)$ can be a proportion α_p of the average yearly income of an individual retiring at time s. Then

$$p_{rep}(s) = \alpha_p \frac{c_s(a_r)}{a_r - a_e}, \text{ with } c_t(a) = \int_{t-a+a_e}^{t-a+a_r} e^{\int_u^t r_s ds} \mathfrak{e}_u(a + u - t) du. \tag{1.10}$$

$c_t(a)$ is the present value of wages earned by an individual's of age a at t.
One can also take instead the average income over the last h years before retirement, then $p_{ret}(s) = \alpha_p \frac{\int_{s-h}^{s} e^{\int_u^s r_v dv} \mathfrak{e}_u(a+u-t) du}{h}$.
• (λ_t) is the indexation rate adjusting pension benefits. The indexation rate takes into account changes in prices or wages, using for example the Consumption Price Index. It can be used to maintain the sustainability of the pension system is case of a demographic shock, such as in [AGDCBPD18].

The adequacy constraint (1.8) yields a liquidity issue for the PAYG system, since the minimum pension amount P_t^{min} is not covered by the workers contributions when $P_t^{min} \geq C_t$. We thus consider a system where the social planner can invest (or borrow) at each time the amount $C_t - P_t$ in (from) a buffer fund, hence sharing the demographic risk between the different generations.

1.3 Buffer fund

Financial market As the demographic (and economic) risk is obviously not fully transferable to financial markets, We an incomplete Itô market , with short rate (r_t) and $d \leq n$ risky assets. The price process $S = (S^i)_{i=1,\dots,d}$ of the risky assets is defined by

$$dS_t^i = S_t^i(\mu_t^i dt + \sum_{j=1}^n \sigma_t^{i,j} dW_t^j), \qquad i = 1, \cdots, d. \tag{1.11}$$

The $d \times n$ volatility matrix (σ_t) is assumed of full rank (that is $(\sigma_t{}^{tr}\sigma_t)$ is invertible, where ${}^{tr}\sigma_t$ is the transposed matrix). The d-dimensional risk premium vector is denoted by (η_t), where $\eta_t = {}^{tr}\sigma_t.(\sigma_t{}^{tr}\sigma_t)^{-1}(\mu_t - r_t\mathbf{1_d})$. We assume that $\int_0^T (|r_t| + \|\eta_t\|^2)dt < \infty$, for any $T > 0$, $\mathbb{P}$.a.s.
η_t is in the vector subspace $\mathcal{R}_t = {}^{tr}\sigma_t(\mathbb{R}^d)$ of $\mathbb{R}^n$. In what follows, for any $\mathbb{R}^n$-valued stochastic process (X_t), we denote by $X^{\mathcal{R}}$ the process such that for all t, $X_t^{\mathcal{R}}$ is the orthogonal projection of X_t onto $\mathcal{R}_t$. Similarly $X_t^{\perp}$ denotes the orthogonal projection of X_t onto the orthogonal vector subspace $\mathcal{R}_t^{\perp}$.

Investment in the buffer fund On this (incomplete) market, the social planner manages a buffer fund that aims to absorb demographic and economic shocks. The dynamics of the fund can be consider as the financial wealth of a single agent with a labor income (or endowment) $C_t = \alpha_c \int_{a_e}^{a_r} \mathfrak{e}_t(a)n(t,a)da$, and consumption process P_t. The amount of money invested in the risky assets at time t is denoted by the d-dimensional vector ϕ_t. Then the self-financing dynamics for the wealth F of the buffer fund with contribution C_t and pension P_t is

$$dF_t = F_t r_t dt + (C_t - P_t)dt + \phi_t \cdot \sigma_t.(dW_t + \eta_t dt).$$

In order to ensure the sustainability of the pension system, a maximum amount of debt is imposed to the buffer fund :

$$\text{(Sustainability constraint)} \quad \forall t \geq 0,\ F_t \geq \mathfrak{K}_t, a.s., \tag{1.12}$$

$\mathfrak{K}_t$ is an adapted predictable process, that can be negative, and $(-\mathfrak{K}_t)$ corresponds to the social planner maximum amount of debt at time t. For instance, $(-\mathfrak{K}_t)$ can represent a proportion of the Gross Domestic Product (GDP), while the no borrowing constraint corresponds to $\mathfrak{K}_t \equiv 0$. We assume throughout the paper that $\mathfrak{K}$ is an Itô process:

$$d\mathfrak{K}_t = \mu_t^{\mathfrak{K}} dt + \delta_t^{\mathfrak{K}} \cdot dW_t. \tag{1.13}$$

Social planner strategy The strategy of the social planner (pensions and investment) is parametrized by (π, ρ) where $P_t = \rho_t P_t^{min}$ and $\pi_t := {}^{tr}\sigma_t \phi_t \in \mathcal{R}_t$. The dynamic dynamic of the buffer with strategy (π, ρ) is thus given by:

$$dF_t^{\pi,\rho} = F_t^{\pi,\rho} r_t dt + (C_t - \rho_t P_t^{min})dt + \pi_t \cdot (dW_t + \eta_t dt). \tag{1.14}$$

Observe that the previous equation is not in a multiplicative form since the value of the fund F can be negative.

Definition 1.1 (Admissible strategy)**.** *An $\mathbb{F}$-adapted strategy (π, ρ) is said to be admissible if and only if*

- $\pi_t \in \mathcal{R}_t, \quad \forall\, t \geq 0$ *$\mathbb{P}$-a.s.*
- $\int_0^t \left(|C_s - \rho_s P_s^{min}| + \|\pi_s\|^2\right) ds < \infty, \quad \forall\, t \geq 0$ *$\mathbb{P}$-a.s.*
- $\rho_t \geq 1$ *(adequacy) and* $F_t^{\pi,\rho} \geq \mathfrak{K}_t$ *(sustainability)* $\quad \forall\, t \geq 0$ *$\mathbb{P}$-a.s.*

The set of all admissible strategies (π, ρ) is denoted $\mathcal{A}$.

Lastly, we introduce the class of the so-called state price density processes (taking into account the discount factor), also called discounted pricing kernels. The discounted pricing kernels Y are characterized by the property that for any admissible strategy (π, ρ), the current wealth plus the cumulative pension minus the cumulative consumption, all discounted by Y(that is $Y_t F_t^{\pi,\rho} + \int_0^t (\rho_s P_s^{min} - C_s) Y_s ds$) is a local martingale (see [EKHM22] for financial and economic viewpoints on the discounted pricing kernels). Discounted pricing kernels are positive Itô-semimartingale with the following dynamics characterized by an orthogonal volatility component $\nu_t \in \mathcal{R}_t^{\perp}$.

Definition 1.2. *(State price density process/discounted pricing kernel). A positive Itô semi-martingale Y^ν is called an admissible state price density process (or discounted pricing kernel) if it has the differential decomposition*

$$Y^\nu = Y^\nu[-r_t dt + (\nu_t - \eta_t) \cdot dW_t], \quad \nu_t \in \mathcal{R}_t^{\perp},\ Y_0^\nu = Y_0. \tag{1.15}$$

2 The social planner's dynamic decision criterion

The pension allocation and the investment in the buffer fund is decided by a social planner. Her decision are taken upon a preference criterion that should take into account present and future generations. Since the social planner has to aggregate the heterogeneous preferences of the different cohorts, her utility criteria is necessarily complex.
Moreover, due to the long-term characteristics of pension schemes, the decision criteria should be adapted to the non-stationary demographic, economic and financial environment in order to provide a consistent strategy in the long run. For both of these aspects, it has been shown that dynamic utility provides a flexible framework to handle this heterogeneity and to propose long-term policies coherent in time ([EKHM17],[EKHM18]).

2.1 Definition of Dynamic Utilities

Dynamic utilities extend the standard notion of deterministic utilities, by allowing the utility criterion to be dynamically adjusted to the available information represented by the filtration $(\mathcal{F}_t)_{t\geq 0}$. A dynamic utility U is a then collection of random utility functions $\{U(t,\omega,z)\}$ whose temporal evolution is "updated" in accordance with the new information $(\mathcal{F}_t)$, starting from an initial utility function $u(z) = U(0,z)$ (which is deterministic if $\mathcal{F}_0$ is trivial). Throughout the paper, we adopt the convention of small letters for deterministic utilities and capital letters for stochastic utilities. The specificity of this paper lies in the sustainability and adequacy constraints for the buffer fund and the pension process, which will be taken into account in the definition domain of dynamic utilities. In the following, we give a precise definition of this extension of dynamic utilities on stochastic domains.

Definition 2.1 (Dynamic Utility on stochastic domain)**.**
A dynamic utility defined on a random domain $U = (U(t,z,\omega))$ is a collection of random utility functions defined on a stochastic domain $\mathcal{D}_U := \{(t,z,\omega), z \geq X_t(\omega)\}$ such that
(i) (X_t) *is an $\mathbb{F}$-adapted process.*
(ii) *For all $t \geq 0$, for all $z \geq X_t$, $U(t,z)$ is $\mathcal{F}_t$-adapted.*
(iii) *There exists $N \in \mathcal{F}$ with $\mathbb{P}(N) = 0$, such that for any $\omega \in N^c$, and for any $t \geq 0$, the functions $z \in [X_t(\omega), \infty[\mapsto U(t,z,\omega)$ are nonnegative, strictly concave increasing functions of class $\mathcal{C}^2$ on $]X_t(\omega), \infty[$.*
(iv) *Inada conditions:* $\lim_{z\to X_t(\omega)} U_z(t,z,\omega) = +\infty$ *and* $\lim_{z\to+\infty} U_z(t,z,\omega) = 0, \quad \forall t \geq 0$ $\mathbb{P}$-*a.s.*

As for a standard utility function, the risk aversion coefficient $\mathrm{R_A}(U)$ is measured by the ratio $\mathrm{R_A}(U)(t,z) = -U_{zz}(t,z)/U_z(t,z)$ and the relative risk aversion by $\mathrm{R^r_A}(U)(t,z) = z\,\mathrm{R_A}(U)(t,z)$. Note that for dynamic utilities, $\mathrm{R_A}$ and $\mathrm{R^r_A}$ are random coefficients.

Remark 2.2. *Note that the Inada condition in X_t implies that the absolute risk aversion coefficient explodes in X_t:*

$$\lim_{z\to X_t(\omega)} R_A(U)(t,z) = \lim_{z\to X_t(\omega)} -\frac{U_{zz}(t,z)}{U_z(t,z)} = +\infty,\ \mathbb{P}\text{-}a.s\ . \tag{2.1}$$

Indeed, denoting $f(t,z) := -\frac{U_{zz}(t,z)}{U_z(t,z)}$, we have that $U_z(t,z) = U_z(t,z_0)\exp(\int_z^{z_0} f(t,u)du)$ for $z_0 > z$ greater than X_t. Then, for a fixed z_0, the Inada condition $\lim_{z\to X_t(\omega)} U_z(t,z) = +\infty$ implies $\lim_{z\to X_t(\omega)} \int_z^{z_0} f(t,u)du = +\infty$ and therefore $\lim_{z\to X_t(\omega)} f(t,z) = +\infty$.

The Fenchel-Legendre convex conjugate of a dynamic utility U defined on a stochastic domain $\mathcal{D}_U := \{(\omega,t,z), z \geq X_t(\omega)\}$ is denoted by $\tilde{U}$, where $\tilde{U}$ satisfies

$$\tilde{U}(t,y) = \sup_{z\geq X_t(\omega)} \big(U(t,z) - yz\big), \quad y \in \mathbb{R}^+.$$

In particular, $\tilde{U}(t,y) \geq U(t,z) - yz$ and the maximum is attained at $U_z(t,z) = y$. Note that $\tilde{U}$ is defined on $\mathbb{R}^+ \times \mathbb{R}^+$ thanks to Inada conditions on U. $\tilde{U}$ is twice continuously y-differentiable, strictly convex, strictly decreasing. Moreover, the marginal utility U_z verifies $U_z^{-1}(t,y) = -\tilde{U}_y(t,y)$; $\tilde{U}(t,y) = U\big(t, -\tilde{U}(t,y)\big) + \tilde{U}_y(t,y)\,y$, and $U(t,z) = \tilde{U}\big(t, U_z(t,z)\big) + zU_z(t,z)$.

Dynamic CRRA utilities Deterministic Constant Relative Risk Aversion (CRRA) utilities is the standard framework in the economic literature, as explained in [Wak08]. They belong to the class of deterministic Hyperbolic Absolute Risk Aversion (HARA) utility (see e.g. the seminal paper of Merton [Mer75], or Kingston and Thorp [KT05]) characterized by an hyperbolic absolute risk aversion coefficient:

$$\mathrm{R_A}(u)(z) = -\frac{u_{zz}(z)}{u_z(z)} = \frac{1}{az+b}, \quad \text{with } a > 1 \text{ and } b \in \mathbb{R}. \tag{2.2}$$

Integrating (2.2) gives that $u(z) = \frac{(z+b/a)^{1-\theta}}{1-\theta}$, defined for $z > -b/a$, and where $\theta = 1/a$. The case $b = 0$ corresponds to CRRA utility (also called power utilities) with constant relative risk aversion θ. Hereafter, we extend the notion of deterministic CRRA utilities to dynamic ones, still with deterministic θ.

Definition 2.3. *Dynamic CRRA utility $U^{(\theta)}(t,z)$, with $\theta \in]0,1[$, are defined by*

$$U^{(\theta)}(t,z) := Z_t^u \frac{(z - X_t)^{1-\theta}}{1-\theta}, \quad \textit{for } z \geq X_t \tag{2.3}$$

where X_t is a stochastic process and Z_t^u is a positive stochastic coefficient reflecting the random evolution of the time preferences.

For instance X_t can represent a borrowing constraint of the social planner. Dynamic CRRA utilities satisfy Inada conditions :

$$\lim_{z\to\infty} U_z^{(\theta)}(t,z) = 0 \text{ and } \lim_{z\to X_t(\omega)} U_z^{(\theta)}(t,z) = \infty, \quad \forall t,\ \mathbb{P}\,a.s..$$

2.2 Buffer fund and pensioners dynamic utilities

The social planner's preference process is defined as $U(t,F_t) + \int_0^t V(s,\rho_s)ds$, where U is the buffer fund dynamic utility, and V is the aggregate dynamic utility of the pensioners. The sustainability constraint (1.12) and the adequacy constraint (1.8) are translated into stochastic domains for both U and V. The supermartingale property induced by the dynamic programming principle will be translated into condition on local characteristics of U. This explains why we assume stronger regularity conditions on U than on V.

Definition 2.4 (Buffer fund utility)**.** *The buffer fund utility U is a dynamic utility with domain $\mathcal{D}_U = \{(\omega,t,z),\ z \geq \mathfrak{K}_t\}$, where $(\mathfrak{K}_t)$ is the sustainability bound satisfying (1.13) and with $U(t,\cdot)$ of class $\mathcal{C}^{3,\delta}$, that is of class $\mathcal{C}^3$ with U_{zzz} δ-Hölder, $\delta \in]0,1[$.*

The preference process of a pensioner of age a at time t, is defined by $\bar{v}(t,a,p_t(a))$, with $p_t(a)$ then pension amount, and $\bar{v}$ a dynamic utility depending of the pensioner's age and taking into account uncertain future changes in the pensioners' preferences.
The social planner aggregates preferences of all pensioners to obtain the aggregated pensioners' dynamic utility defined by $\int_{a_r}^{\infty}\bar{v}(t,a,p_t(a))\omega_t(a)n(t,a)da$, with $\omega_t(a)$ the weight given to a pensioner of age a at time t. For instance, the social planner can take into account the actuarial fairness by giving more weight to pensioners who contributed more.
Recalling that $p_t(a) = p_t^{min}(a)\rho_t$, we can define a weighted dynamic utility v applied to ρ_t, with

$$v(t,a,\rho) = \omega_t(a)\bar{v}(t,a,p_t^{min}(a)\rho). \tag{2.4}$$

In the following, we refer to v as a pensioner dynamics utility from the viewpoint of the social planner.

Definition 2.5 (Pensioners' utility)**.** *We assume that a pensioner at time t and of age a has the dynamic utility $v(t,a,\cdot)$, defined as in (2.4) on a domain $[\underline{\rho}_t,+\infty[$, with $\underline{\rho}_t \le 1$. The aggregated utility of pensioners is defined by:*

$$V(t,\rho) = \int_{a_r}^{\infty} v(t,a,\rho)n(t,a)da. \tag{2.5}$$

Example 1 We come back to the case where each pensioner receives the same pension amount $p_t = \rho_t p_t^{min} \ge p_t^{min}$ and has the same dynamic utility $\bar{v}(t,p)$. In order to take into account actuarial fairness, the social planner can aggregate preferences of all living pensioners by weighting their utility by their past contributions $\alpha_c c_t(a)$ (in the same spirit of [GG12]), introduced in (1.10). Then,

$$v(t,a,\rho) = \alpha_c c_t(a)\bar{v}(t,\rho\, p_t^{min}),$$

and the aggregate utility from pensions, taking into account the weight of each pensioners' generation, is

$$V(t,\rho) = \int_{a_r}^{\infty} \bar{v}(t,\rho\, p_t^{min})\alpha_c c_t(a)n(t,a)da = \omega_t^r\bar{v}(t,\rho\, p_t^{min}), \tag{2.6}$$

with the total weight of all pensioners living at time t given by

$$\omega_t^r = \alpha_c \int_{a_r}^{\infty} c_t(a)n(t,a)da. \tag{2.7}$$

Example 2: In the second example of age-dependent pension, $p_t(a) = p_t^{min}(a)\rho_t$ with $p_t^{min}(a) = p_{ret}(a_r+t-a)e^{\int_{a_r+t-a}^{t}\lambda_s ds}$.We further assume that the pensioners utility $\bar{v}(t,a,p)$ may depend on their age, so that $v(s,a,\rho) = \bar{v}(s,a,\rho\, p_s^{min}(a))$. Then the actuarial fairness criteria is taken into account via the choice of initial pension amount $p_{ret}(s)$, which increases with the past contribution of the individual retiring at time s. Thus, the social planner may aggregate the pensioners' utility without using any correcting weight. Then

$$V(t,\rho) = \int_{a_r}^{\infty} \bar{v}(t,a,\rho\, p_{ret}(a_r+t-a)e^{\int_{a_r+t-a}^{t}\lambda_u du})\, n(t,a)da \tag{2.8}$$

We introduce the following assumption on the rates of increase of the marginal dual utilities $\tilde{V}_\rho$ and $\tilde{U}_z$, that will play a role in proving the existence of an admissible optimal portfolio in Theorem 3.5.

Assumption 2.6. *We assume the existence of a locally integrable process B such that*

$$|\tilde{V}_\rho(t,z) - \tilde{V}_\rho(t,z')| \le B_t\, |\tilde{U}_z(t,z) - \tilde{U}_z(t,z')|, \quad \textit{for } t \ge 0,\ z > 0,\ z' > 0. \tag{2.9}$$

2.3 Consistency Property

Social planner optimization problem The social planner has to manage a tradeoff between the pension payed to the pensioners and the fund that constitutes reserves for the future generations, among all the admissible strategies satisfying the sustainability and adequacy. In the usual setting, the optimization program is posed backward. It is formulated on a given horizon T_H, and is written at time $t=0$ as follows (given $F_0 = x$):

$$\mathcal{U}(0,x) := \sup_{(\pi,\rho)\in\mathcal{A}} \mathbb{E}\big(u(T_H, F^{\pi,\rho}_{T_H}) + \int_0^{T_H} V(t,\rho_t)dt\big). \tag{2.10}$$

In the backward formulation, the utilities u of terminal wealth (at T_H) and V of pension rate are given. In our context of intergenerational risk-sharing for pensions, fixing a (long-term) time-horizon T_H and even more a utility function $u(T_H,.)$ seems artificial. Extending the optimization program and the optimal strategy to a horizon larger to T_H, in a time-consistent way, is also a difficult issue. In order to ensure consistency across time and generations, the social planner should be able to identify which "terminal" criterion $U(T,.)$ should be considered at any intermediate date $T \le T_H$, while still leading to the same optimal strategy and the same value $\mathcal{U}(0,x)$, that is satisfying

$$\text{for any } T \le T_H,\ \mathcal{U}(0,x) = \sup_{(\pi,\rho)\in\mathcal{A}} \mathbb{E}\big(U(T, F^{\pi,\rho}_T) + \textstyle\int_0^T V(t,\rho_t)dt\big).$$

Under regularity assumptions, this criterion is given by the "value function" $\mathcal{U}(T,z)$ given the wealth $F_T = z$ at time T

$$\mathcal{U}(T,z) = \text{“}\sup_{(\pi,\rho)\in\mathcal{A}}\text{”}\, \mathbb{E}\Big(u(T_H, F^{\pi,\rho}_{T_H}(T,z)) + \int_T^{T_H} V(s,\rho_s)ds | F_T = z\Big),\ a.s.. \tag{2.11}$$

This time-consistency translates into a martingale property of the preference process $\mathcal{U}(t, F^{(\pi,\rho)}_t) + \int_0^t V(s,\rho_s)ds$ along the optimal strategy. This property, known as the dynamic programming principle, is the main tool in the theory of stochastic control, see Davis [Dav79] or El Karoui [EK81]. In this backward setting, $\mathcal{U}(T_H,.) = u(T_H,.)$ is given, and the unknown

is the optimal strategy (F^*, ρ^*) as well as $\mathcal{U}(t,.)$, also called "indirect" utility, possibly stochastic. Nevertheless, $\mathcal{U}$ is difficult to compute (even if $u(T_H,.)$ is given as a simple deterministic function), and it is even not trivial to prove that $\mathcal{U}$ defined by (2.11) is indeed concave.
In the forward setting, there is no intrinsic time-horizon T_H and this is the initial utility $\mathcal{U}(0,.)$ which is given. This means that forward utilities differ from backward utilities by their boundary conditions, both satisfying a dynamic programming principle, also called consistency given the constraints set $\mathcal{A}$.

Consistency and optimal strategy The satisfaction provided by an admissible strategy $(\pi, \rho) \in \mathcal{A}$ is measured by the dynamic criterion $U(t, F_t^{\pi,\rho}) + \int_0^t V(s, \rho_s)ds.$ that is assumed to satisfy a dynamic programming principle.

Definition 2.7 (Consistent dynamic utility). *Let (U, V) be a dynamic utility system with admissible strategies set $\mathcal{A}$. The utility system (U, V) is said to be consistent, if*
(i) *For any admissible strategies $(\pi, \rho) \in \mathcal{A}$, the preference process $(U(t, F_t^{\pi,\rho}) + \int_0^t V(s, \rho_s)ds)$ is a non-negative supermartingale.*
(ii) *There exists an* optimal *strategy $(\pi^*, \rho^*) \in \mathcal{A}$, binding the constraints, in the sense that the optimal preference process $(U(t, F_t^{\pi^*,\rho^*}) + \int_0^t V(s, \rho_s^*)ds)$ is a martingale.*

Under regularity assumptions, the value function $(\mathcal{U}(t, z), V(t, \rho))$ of the classical optimization problem is an example of consistent utility system, defined from its terminal condition $\mathcal{U}(T_H, z) = u(z)$ (see [EKHM18] and [EKHM22] for a general discussion between the forward and the backward viewpoints of utility functions).

2.4 Semimartingale dynamic utility

In order to study the preference process $(U(t, F_t^{\pi,\rho}) + \int_0^t V(s, \rho_s)ds)$, we assume in the following that the buffer fund dynamic utility U defined in 2.4 is an Itô random field:

$$dU(t, z) = \beta(t, z)dt + \gamma(t, z).dW_t, \quad z \geq \mathfrak{K}_t, \tag{2.12}$$

with $\beta(t, z)$ the drift random field and $\gamma(t, z)$ the multivariate diffusion random field. Since the domain of the buffer fund utility U is time varying, its dynamics is defined more precisely by introducing the shifted utility $\bar{U}$ with fixed domain $\mathbb{R}^+ \times \mathbb{R}^+$:

$$\bar{U}(t, z) := U(t, z + \mathfrak{K}_t), \quad \forall (t, z) \in \mathbb{R}^+ \times \mathbb{R}^+, \tag{2.13}$$

and with local characteristics denoted by $(\bar{\beta}, \bar{\gamma})$. Obviously U is a dynamic utility on $\mathcal{D}_U$ if and only if $\bar{U}$ is a dynamic utility on $\mathbb{R}^+ \times \mathbb{R}^+$. These semimartingale dynamic utilities have been studied in details in [EKM13]. An important point is the connection between the regularity of the dynamic utility $\bar{U}$ and that of its local characteristics $(\bar{\beta}, \bar{\gamma})$. If $\bar{U}$ is of class $\mathcal{C}^{2,\delta}$ then its

characteristics $\bar{\beta}$ and $\bar{\gamma}$ are of class $\mathcal{C}^{2,\varepsilon}$ for all $0<\varepsilon<\delta$ and conversely if $\bar{\beta}$ and $\bar{\gamma}$ are in the class $\mathcal{C}^{2,\delta}$ then U is in $\mathcal{C}^{2,\varepsilon}$ for all $0<\varepsilon<\delta$ ([Kun97], [EKM13]). Another point are conditions on $(\bar{\beta},\bar{\gamma})$ for $\bar{U}$ to be a dynamic utility. Indeed, in the absence of general comparison results for stochastic integrals, it is not straightforward to obtain conditions on the local characteristics $(\bar{\beta},\bar{\gamma})$ such that the process $\bar{U}(t,z)=\bar{U}(t,0)+\int_0^t \bar{\beta}(t,s)ds+\bar{\gamma}(s,z).dW_s$ is increasing and concave.
We recall sufficient assumptions below:

Assumption 2.8. *Let $\bar{U}(t,z)=U(t,z+\mathfrak{K}_t)$ the shifted buffer fund utility. We assume that there exists random bounds (B^1_t) and (ζ_t) such that a.s. $\int_0^T B^1_t dt<\infty$ and $\int_0^T \zeta^2_t dt<\infty$ for any T, and such that for any $z>0$:*

$$\begin{cases} |\bar{\beta}_z(t,z)|\leq B^1_t\,\bar{U}_z(t,z), & |\bar{\beta}_{zz}(t,z)|\leq B^1_t\,|\bar{U}_{zz}(t,z)| \quad (2.14)\\ \|\bar{\gamma}_z(t,z)\|\leq \zeta_t\,\bar{U}_z(t,z), & \|\bar{\gamma}_{zz}(t,z)\|\leq \zeta_t\,|\bar{U}_{zz}(t,z)| \quad (2.15)\end{cases}$$

Under Assumption 2.8, $\bar{U}$ is a well-defined dynamic utility by Corollary 1.3 in [EKM13].

Itô-Ventzel's formula The link between the local characteristics (β,γ) of U and $(\bar{\beta},\bar{\gamma})$ of $\bar{U}$ are deduced from the Itô-Ventzel formula, which is a generalization of the Itô formula in the case where the function is itself a random field.
The Itô-Ventzel formula gives the decomposition of a compound random field $U(t,X_t)$ for $U(t,z)=u(0,z)+\int_0^t\beta(s,z)ds+\int_0^t\gamma(s,z).dW_s$ regular enough (of class $\mathcal{C}^{2,\delta}$, with $\delta\in]0,1[$) and any Itô semimartingale X. This decomposition is the sum of three terms: the first one is the "differential in t" of U, the second one is the classic Itô's formula (without differentiation in time) and the third one is the infinitesimal covariation between the martingale part of U_z and the martingale part of X, all these terms being taken in X_t.

$$\begin{aligned} dU(t,X_t) &= \big(\beta(t,X_t)\,dt+\gamma(t,X_t).dW_t\big) \quad (2.16)\\ &+ \Big(U_z(t,X_t)dX_t+\frac{1}{2}U_{zz}(t,X_t)d<X,X>_t\Big)+\big(<\gamma_z(t,X_t)\cdot dW_t,dX_t>\big).\end{aligned}$$

Applying the Itô-Ventzell formula to $\bar{U}(t,z-\mathfrak{K}_t)$ yields the following result.

Proposition 2.9. *Recall that the bound $\mathfrak{K}_t$ is an Itô process with dynamics $d\mathfrak{K}_t=\mu^{\mathfrak{K}}_t dt+\delta^{\mathfrak{K}}_t\cdot dW_t$. Under Assumption 2.8, U is a dynamic utility on the domain $\mathcal{D}_U=\{(\omega,t,z),\ z\geq\mathfrak{K}_t\}$, with local characteristics*

$$\begin{cases}\beta(t,z)=\bar{\beta}(t,z-\mathfrak{K}_t)-U_z(t,z)\mu^{\mathfrak{K}}_t-\frac{1}{2}U_{zz}(t,z)\|\delta^{\mathfrak{K}}_t\|^2-\gamma_z(t,z)\cdot\delta^{\mathfrak{K}}_t \quad (2.17)\\ \gamma(t,z)=\bar{\gamma}(t,z-\mathfrak{K}_t)-U_z(t,z)\delta^{\mathfrak{K}}_t,\quad \forall z\geq\mathfrak{K}_t. \quad (2.18)\end{cases}$$

3 Optimal PAYG pension policies

Let us recall that a pensioner of age a at time t receives pension $p(t,a) = p_t^{min}(a)\rho_t \geq p_t^{min}(a)$. The buffer fund in which the social planner can borrow/invest has the dynamics (1.13) given by:

$$dF_t^{\pi,\rho} = F_t^{\pi,\rho} r_t dt + (C_t - \rho_t P_t^{min})dt + \pi_t \cdot (dW_t + \eta_t dt),$$

with (C_t) the contribution process, $(P_t) = (\rho_t P_t^{\min})$ the pension process introduced in (1.7), and (π_t) the investment strategy in the incomplete market. The aggregated utility of pensioners is given by $V(t,\rho) = \int_{a_r}^{\infty} v(t,a,\rho)n(t,a)da$ (see (2.5) and subsequent examples). The first aim is to characterize the buffer fund utility U of the social planner, such that the preference criteria (U,V) is consistent, and then to determine the admissible strategy $(\pi,\rho) \in \mathcal{A}$ optimizing the dynamic criterion $U(t, F_t^{\pi,\rho}) + \int_0^t V(s,\rho_s)ds$.

3.1 Consistency SPDE

The Itô-Ventzel's formula allows us to transform the supermartingale property implied by the consistency condition into conditions on the differential characteristics of the utility process U. For standard deterministic utility functions, the infinitesimal counterpart of the dynamic programming principle is a nonlinear Partial Differential Equation (PDE), called dynamic programming equation or Hamilton-Jacobi-Bellman (HJB) equation. In the framework of dynamic utility, the consistency characterization is given in terms of an HJB Stochastic Partial Differential Equation (SPDE). The presence of pensions to be paid impacts this SPDE in a non-linear way, the non-linear factor involving the utility of pensioners V. Note that the utility U and V are not of the same nature : the consistency is conveyed by U, which requires then stronger regularity conditions on U than on V. This HJB-SPDE provides a constraint on the drift β of U, as explained below.

Candidate to be the optimal strategy Applying Itô-Ventzel's formula to the preference process $Z_t^{\pi,\rho} := \int_0^t V(s,\rho_s)ds + U(t, F_t^{\pi,\rho})$ with yields

$$\begin{aligned} dZ_t^{\pi,\rho} &= \big(\beta(t,F_t^{\pi,\rho}) + U_z(t,F_t^{\pi,\rho})(F_t^{\pi,\rho} r_t + C_t)\big)dt \\ &+ \big(\gamma(t,F_t^{\pi,\rho}) + U_z(t,F_t^{\pi,\rho})\pi_t\big) \cdot dW_t \\ &+ \big(\mathcal{P}(t,F_t^{\pi,\rho},\rho_t) + \mathcal{Q}(t,F_t^{\pi,\rho},\pi_t)\big)dt. \end{aligned}$$

with
$$\mathcal{P}(t,z,\rho) := V(t,\rho) - U_z(t,z)P_t^{min}\rho \tag{3.1}$$
$$\mathcal{Q}(t,z,\pi) := \frac{1}{2}U_{zz}(t,z)||\pi_t||^2 + \pi_t \cdot (\gamma_z(t,z) + U_z(t,z)\eta_t). \tag{3.2}$$

A natural candidate for optimal policy (π^*,ρ^*) are processes which maximize the drift of the preference process $Z^{\pi,\rho}$. Thus, π^* should maximize $\mathcal{Q}(t,z,\cdot)$ and ρ^* should maximize

$\mathcal{P}(t,z,\cdot)$ on $[1,+\infty]$, leading to

$$\pi_t^*(F_t^*) = -\frac{\gamma_z^{\mathcal{R}}(t,F_t^*) + U_z(t,F_t^*)\eta_t}{U_{zz}(t,F_t^*)} \quad \text{and} \quad \rho_t^*(F_t^*) = V_\rho^{-1}\big(t, P_t^{min} U_z(t,F_t^*)\big) \vee 1. \tag{3.3}$$

Note that $\mathcal{Q}(t,z,\pi_t^*) = -\frac{1}{2}U_{zz}(t,z)\|\pi_t^*\|^2$.
With a slight abuse of notation, we use interchangeably π_t^* (resp. ρ_t^*) or $\pi_t^*(F_t^*)$ (resp. $\rho_t^*(F_t^*)$).
To alleviate the notations, we introduce the optimal pension process without minimum guarantee, denoted ρ^f (free constraints).

Definition 3.1. *The maximizer of the operator $\mathcal{P}(t,z,\rho) := V(t,\rho) - U_z(t,z)P_t^{min}\rho$ for $\rho \in \mathbb{R}^+$ is denoted*

$$\rho_t^f(z) = V_\rho^{-1}\big(t, P_t^{min} U_z(t,z)\big). \tag{3.4}$$

Remark that $\mathcal{P}(t,z,\rho_t^f(z)) = \tilde{V}(t, P_t^{min} U_z(t,z))$.

Remark 3.2. *Note that if U satisfies the Inada condition at $\mathfrak{K}$, then v also satisfy the Inada condition at $\underline{\rho}$,in order to ensure that the optimal pension is well defined (namely the quantity $V_\rho^{-1}\big(t, P_t^{min} U_z(t,z)\big)$). Remark also that Inada conditions for U and v at $+\infty$ can be relaxed into the following condition $\lim_{\rho\to+\infty} V_\rho(t,\rho) \le P_t^{min} \lim_{z\to+\infty} U_z(t,z)$.*

Consistency condition on the drift β of U If the candidate (π^*,ρ^*) is indeed the optimal strategy, then to satisfy the time consistency, the drift of the preference process $Z^{\pi,\rho}$ should be nonpositive for any admissible strategy (π,ρ) and equal to zero for the optimal stragegy (π^*,ρ^*). This leads to the following sufficient condition on the drift of U

$$\begin{aligned} \beta(t,z) &= -U_z(t,z)(zr_t + C_t) - \mathcal{Q}(t,z,\pi^*) - \mathcal{P}(t,z,\rho_t^*(z)) \\ &= -U_z(t,z)\big(zr_t + C_t - P_t^{min}\rho^*\big) + \frac{1}{2}U_{zz}(t,z)\|\pi_t^*(z)\|^2 - V(t,\rho_t^*(z)). \end{aligned} \tag{3.5}$$

3.2 Main results

We gather here the main results the paper, proofs and examples being postponed in the next subsections. The first below result shows that under the consistency HJB condition (3.5), the bound $\mathfrak{K}$ shifting the utility U is necessarily a buffer fund. This is an interesting new result.

Theorem 3.3. *Let U be the buffer fund utility introduced in 2.4 and verifying Assumption 2.8, and V the aggregated pensioners' utility, verifying Assumption 2.6. Assume that the drift β of U satisfies the HJB constraint*

$$\beta(t,z) = -U_z(t,z)\big(zr_t + C_t - P_t^{min}\rho_t^*(z)\big) + \frac{1}{2}U_{zz}(t,z)\|\pi_t^*(z)\|^2 - V(t,\rho_t^*(z)). \tag{3.5}$$

Then the sustainability bound $\mathfrak{K}$ is necessarily a buffer fund receiving the contribution C and paying the minimal pension amount P^{min}, that is:

$$d\mathfrak{K}_t = (\mathfrak{K}_t r_t + C_t - P_t^{min})dt + \delta_t^{\mathfrak{K}} \cdot (dW_t + \eta_t dt), \;\; \delta^{\mathfrak{K}} \in \mathcal{R}. \tag{3.6}$$

The proof is postponed to Section 3.3.

Remark 3.4. Theorem 3.3 states that if the utility system (U, V) is consistent, then the shift $\mathfrak{K}_t$ of the utility U is necessarily a buffer fund with minimal pensions. Nevertheless, in the original problem formulation and Definition 2.4 of U, $\mathfrak{K}_t$ is not assumed to be a buffer fund itself: for instance if $\mathfrak{K}$ is an index following the GDP, there is no reason why it will follow dynamics (3.6). Therefore, to satisfy the consistency constraint, the dynamic utility should be shifted not with the sustainability constraint $\mathfrak{K}_t$, itself but with a buffer fund process (X_t) that super-replicates (in a pathwise way) $(\mathfrak{K}_t)$ that is : $X_t \geq \mathfrak{K}_t$, for all t, $\mathbb{P}$ a.s. This means that the sustainability constraint is transformed into a stronger one: $F_t \geq X_t$ $(\geq \mathfrak{K}_t)$, for all t, $\mathbb{P}$ a.s.. The problem is equivalent to searching for a self-financing portfolio (without contributions and pensions) X' super-replicating (pathwise) the process $B_t := \mathfrak{K}_t + \int_0^t (P_s^{min} - C_s)ds$. For example, it may be relevant to choose the "minimal" super-replicating self-financing portfolio (X_t') (if it exists) in the following sense:

$$X_0^* := \inf\{X_0 \geq \mathfrak{K}_0 \; s.t. \; \exists \pi' \in \mathcal{R} \;\text{ satisfying }\; X_t' := X_0 + \int_0^t r_s X_s' ds + \pi_s' \cdot (dW_s + \eta_s ds) \geq B_t, \; dt \otimes d\mathbb{P} a.s.\}$$

The existence of a super-replicating self-financing portfolio is not guaranteed, especially in our context of incomplete market, in which demographic risk can not be completely hedged by financial assets. Applying Theorem 5.12 of Karatzas and Kou [KK98], a sufficient existence condition on $[0, T]$ is

$$\sup_{\nu \in \mathcal{R}^{\perp}} \sup_{\tau \in \mathcal{T}_{[0,T]}} \mathbb{E}\left(Y_\tau^\nu \left(\mathfrak{K}_\tau + \int_0^\tau (P_s^{min} - C_s)ds\right)^+ \right) < \infty, \quad \forall T \geq 0, \tag{3.7}$$

where $\mathcal{T}_{[0,T]}$ is the class of stopping times τ with values in the interval $[0, T]$ and Y^ν the state price density process (1.15). Note that this supremum corresponds to X_0^* which is the super-replicating price of (B_t).
In the backward framework, El Karoui and Jeanblanc [EKJP98], or He and Pagès [HP93] consider a complete market, which ensures the existence of a super-replicating portfolio. In incomplet market, an analoguous assumption as (3.7) is needed in Mostovyi and Sirbu [MS20] (Assumption 2.5), to ensures the existence of a super-replicating portfolio (see Lemma 3.2 [MS20]).

The following theorem shows that the policy given by (3.3) is indeed the optimal strategy, and that the corresponding buffer fund satisfies the sustainability constraint (1.12).

Theorem 3.5. *Let U be the buffer fund utility verifying Assumptions 2.8, and V be the aggregated pensioners' utility, verifying Assumption 2.6. Assume $F_0 > \mathfrak{K}_0$ and that the drift*

β of U satisfies the HJB constraint (3.5):

$$\beta(t,z) = -U_z(t,z)\big(zr_t + C_t - P_t^{min}\rho_t^*(z)\big) + \frac{1}{2}U_{zz}(t,z)\|\pi_t^*(z)\|^2 - V(t,\rho_t^*(z)),$$

$$\text{with} \begin{cases} \pi_t^*(z) = -\dfrac{\gamma_z^{\mathcal{R}}(t,z) + U_z(t,z)\eta_t}{U_{zz}(t,z)}, & (3.8)\\ \rho_t^*(z) = V_\rho^{-1}\big(t, P_t^{min}U_z(t,z)\big) \vee 1. & (3.9)\end{cases}$$

Then the portfolio/pension plan (π^*, ρ^*) is the optimal strategy. In particular, the buffer fund F^* following the investment strategy $\pi_t^* = \pi_t^*(F_t^*)$ and paying the pension amount $\rho_t^* = \rho_t^*(F_t^*)$ is strictly greater than $\mathfrak{K}_t$ and satisfies the dynamics

$$\begin{cases} dF_t^* = \mu_t^*(F_t^*)dt + \pi_t^*(F_t^*).dW_t, & (3.10)\\ \mu_t^*(z) := zr_t + C_t - P_t^{min}\rho_t^*(z) + \pi_t^*(z)\eta_t. \end{cases}$$

The proof is postponed to Section 3.4.
Since $V_\rho^{-1}(t,\cdot)$ and $U_z(t,\cdot)$ are decreasing functionals, the optimal pension $p_t^*(a) = \rho_t^* p_t^{min}(a)$ is increasing in the fund's wealth, which is natural. On the other hand, when minimum pension amount P_t^{min} to be paid increases (for instance due to an increase in the number of retirees), $p_t^*(a)$ decreases.

Corollary 3.6. *Under the assumptions and notations of Theorem 3.5,*

$$\lim_{z\to\mathfrak{K}_t} \pi_t^*(z) = \delta_t^{\mathfrak{K}}, \quad \lim_{z\to\mathfrak{K}_t} \rho_t^*(z) = 1, \quad \forall t \geq 0\ \mathbb{P}-a.s. \tag{3.11}$$

This means that the optimal strategy converges to the "minimal" buffer fund strategy when F^* tends to the sustainability bound.

Proof. First, using the expression of the optimal portfolio (3.8),

$$\begin{aligned} \|\pi_t^*(z) - \delta_t^{\mathfrak{K}}\| &= \Big\|\frac{\gamma_z^{\mathcal{R}}(t,z) + U_z(t,z)\eta_t}{U_{zz}(t,z)} + \delta_t^{\mathfrak{K}}\Big\| \\ &\leq \Big\|\frac{\gamma_z^{\mathcal{R}}(t,z) + U_{zz}(t,z)\delta_t^{\mathfrak{K}}}{U_{zz}(t,z)}\Big\| + \|\eta_t\|\frac{U_z(t,z)}{|U_{zz}(t,z)|} \\ &\overset{(2.15)}{\leq} (C_t + \|\eta_t\|)\frac{U_z(t,z)}{|U_{zz}(t,z)|} \overset{z\to\mathfrak{K}_t}{\longrightarrow} 0 \end{aligned}$$

since $\lim_{z\to\mathfrak{K}_t} \frac{U_{zz}(t,z)}{U_z(t,z)} = -\infty$ from Remark 2.2. Finally, by the Inada condition on U and V,

$$\lim_{z\to\mathfrak{K}_t} \rho_t^*(z) \vee 1 = \lim_{z\to\mathfrak{K}_t} V_\rho^{-1}\big(t, P_t^{min}U_z(t,z)\big) \vee 1 = \lim_{\rho\to+\infty} V_\rho^{-1}(t,\rho) \vee 1 = \underline{\rho}_t \vee 1 = 1.$$

□

Example 1 We come back to the first example, in which the pension and individual pensioners' utility $\bar{v}$ do not depend on the age of the pensioner. The social planner attributes the global weight $\omega_t^r = \alpha_c \int_{a_r}^{\infty} c_t(a) n(t,a) da$ to pensioners living at time t, based on their past contributions (see (2.7)). In this example, the aggregate utility of pension is given by (2.6):

$$V(t,\rho) = \int_{a_r}^{\infty} \bar{v}(t, \rho p_t^{min}) \alpha_c c_t(a) n(t,a) da = \omega_t^r \bar{v}(t, \rho p_t^{min}).$$

We have $p_t^{min} V_\rho^{-1}(t,z) = \bar{v}_\rho^{-1}\big(t, \frac{z}{p_t^{min}\omega_t^r}\big)$. Besides, $P_t^{min} = N_t^r p_t^{min}$ where $N_t^r = \int_{a_r}^{\infty} n(t,a)da$ is the number of pensioners at time t. This implies that the optimal pension for each pensioner is

$$p_t^*(z) = p_t^{min} \rho_t^*(z) = \bar{v}_\rho^{-1}\big(t, \frac{N_t^r}{\omega_t^r} U_z(t,z)\big) \vee p_t^{min}. \tag{3.12}$$

The pension amount then increases with the quantity $\frac{\omega_t^r}{N_t^r}$, corresponding to the average individual contribution of a pensioner living at time t.

Example 2 In the second example of age-dependent pension and utility $\bar{v}$, the aggregate utility of pension V is a complex aggregation between cohorts given by (2.8):

$$V(t,\rho) = \int_{a_r}^{\infty} \bar{v}(t, a, \rho\, p_{ret}(a_r + t - a) e^{\int_{a_r+t-a}^{t} \lambda_u du})\, n(t,a) da.$$

In all generality, there is no straightforward formula for V_ρ^{-1} in terms of the $\bar{v}_\rho^{-1}$, except in particular cases of dynamic utilities such as dynamic CRRA utilities (see Section 4).

3.3 Proof of Theorem 3.3

Theorem 3.3 states that in order to satisfy the consistency condition (3.5), the sustainability bound $\mathfrak{K}$ is necessarily a buffer fund receiving the contribution C and paying the minimal pension amount P^{min}. We recall the dynamics of $\mathfrak{K}$ is given by (1.13): $d\mathfrak{K}_t = \mu_t^{\mathfrak{K}} dt + \delta_t^{\mathfrak{K}} \cdot dW_t$. As in Proposition 2.9, we consider hereafter the stochastic utility $\bar{U}(t,z) = U(t, z + \mathfrak{K}_t)$ whose local characteristics are given, by

$$\begin{cases} \beta(t,z) = \bar{\beta}(t, z - \mathfrak{K}_t) - \bar{U}_z(t, z - \mathfrak{K}_t)\mu_t^{\mathfrak{K}} + \frac{1}{2}\bar{U}_{zz}(t, z - \mathfrak{K}_t)||\delta_t^{\mathfrak{K}}||^2 - \bar{\gamma}_z(t, z - \mathfrak{K}_t)\delta_t^{\mathfrak{K}}, \\ \gamma(t,z) = \bar{\gamma}(t, z - \mathfrak{K}_t) - \bar{U}_z(t, z - \mathfrak{K}_t)\delta_t^{\mathfrak{K}}, \end{cases}$$

by the Itô-Ventzel's formula. This combined with the HJB-constraint below,

$$\beta(t,z) = -U_z(t,z)(zr_t + C_t) + \frac{1}{2} U_{zz}(t,z) || \frac{\gamma_z^{\mathcal{R}}(t,z) + U_z(t,z)\eta_t}{U_{zz}(t,z)} ||^2 - V(t, \rho_t^*(z)) + U_z(t,z) P_t^{min} \rho_t^*(z),$$

yields that

$$\begin{aligned}&\bar{\beta}(t,z-\mathfrak{K}_t)-\bar{U}_z(t,z-\mathfrak{K}_t)\mu_t^{\mathfrak{K}}+\frac{1}{2}\bar{U}_{zz}(t,z-\mathfrak{K}_t)||\delta_t^{\mathfrak{K}}||^2-\bar{\gamma}_z(t,z-\mathfrak{K}_t)\cdot\delta_t^{\mathfrak{K}}\\&\quad=-\bar{U}_z(t,z-\mathfrak{K}_t)(zr_t+C_t)-V(t,\rho_t^*(z))+\bar{U}_z(t,z-\mathfrak{K}_t)P_t^{min}\rho_t^*(z)\\&\quad+\frac{1}{2}\bar{U}_{zz}(t,z-\mathfrak{K}_t)||\frac{\bar{\gamma}_z^{\mathcal{R}}(t,z-\mathfrak{K}_t)-\bar{U}_{zz}(t,z-\mathfrak{K}_t)\delta_t^{\mathfrak{K},\mathcal{R}}+\bar{U}_z(t,z-\mathfrak{K}_t)\eta_t}{\bar{U}_{zz}(t,z-\mathfrak{K}_t)}||^2.\end{aligned}$$

Reorganizing the terms, we have (omitting the variables $(t, z - \mathfrak{K}_t)$ to simplify notations):

$$\bar{\beta}+V(t,\rho_t^*(z))=\bar{U}_z\;(\mu_t^{\mathfrak{K}}-zr_t-C_t+P_t^{min}\rho_t^*(z))+\frac{1}{2}\frac{1}{\bar{U}_{zz}}\big(||\bar{\gamma}_z^{\mathcal{R}}-\bar{U}_{zz}\delta_t^{\mathfrak{K},\mathcal{R}}+\bar{U}_z\eta_t||^2-||\bar{U}_{zz}\delta_t^{\mathfrak{K}}||^2\big)+\bar{\gamma}_z\cdot\delta_t^{\mathfrak{K}}.$$

Rewriting the last term as follows,

$$\begin{aligned}&\frac{1}{2}\frac{1}{\bar{U}_{zz}}\big(||\bar{\gamma}_z^{\mathcal{R}}-\bar{U}_{zz}\delta_t^{\mathfrak{K},\mathcal{R}}+\bar{U}_z\eta_t||^2-||\bar{U}_{zz}\delta_t^{\mathfrak{K}}||^2\big)+\bar{\gamma}_z\cdot\delta_t^{\mathfrak{K}}\\&\quad=\frac{1}{2}\frac{1}{\bar{U}_{zz}}\big(||\bar{\gamma}_z^{\mathcal{R}}+\bar{U}_z\eta_t||^2-2(\bar{\gamma}_z^{\mathcal{R}}+\bar{U}_z\eta_t)\cdot\bar{U}_{zz}\delta_t^{\mathfrak{K}}-||\bar{U}_{zz}\delta_t^{\mathfrak{K},\perp}||^2\big)+\bar{\gamma}_z\cdot\delta_t^{\mathfrak{K}}\\&\quad=\frac{1}{2}\frac{1}{\bar{U}_{zz}}\big(||\bar{\gamma}_z^{\mathcal{R}}+\bar{U}_z\eta_t||^2-||\bar{U}_{zz}\delta_t^{\mathfrak{K},\perp}||^2\big)+\bar{\gamma}_z^{\perp}\cdot\delta_t^{\mathfrak{K}}-\bar{U}_z\eta_t\cdot\delta_t^{\mathfrak{K}},\end{aligned}$$

we get that the consistency condition (3.5) is equivalent to

$$\begin{aligned}&\bar{\beta}(t,z-\mathfrak{K}_t)+V(t,\rho_t^*(z))\\=\;&\bar{U}_z\;(\mu_t^{\mathfrak{K}}-zr_t-C_t+P_t^{min}\rho^*(z)-\eta_t\cdot\delta_t^{\mathfrak{K}})+\frac{1}{2}\frac{1}{\bar{U}_{zz}}\big(||\bar{\gamma}_z^{\mathcal{R}}+\bar{U}_z\eta_t||^2-||\bar{U}_{zz}\delta_t^{\mathfrak{K},\perp}||^2\big)+\bar{\gamma}_z^{\perp}\cdot\delta_t^{\mathfrak{K}}\\=\;&\bar{U}_z\Big[\mu_t^{\mathfrak{K}}-zr_t-C_t+P_t^{min}\rho_t^*(z)-\eta_t\cdot\delta_t^{\mathfrak{K}}+\frac{1}{2}\frac{1}{\bar{U}_{zz}\bar{U}_z}\big(||\bar{\gamma}_z^{\mathcal{R}}+\bar{U}_z\eta_t||^2-||\bar{U}_{zz}\delta_t^{\mathfrak{K},\perp}||^2\big)+\frac{\bar{\gamma}_z^{\perp}}{\bar{U}_z}\cdot\delta_t^{\mathfrak{K}}\Big].\end{aligned}$$

Let us analyze the behavior of the left hand size and the right hand size of this identity, when $z \to \mathfrak{K}_t$. We have by 2.4 and 2.5 that $\lim_{z\to\mathfrak{K}_t}\rho_t^*(z)=\lim_{z\to\mathfrak{K}_t}V_p^{-1}\big(t,P_t^{min}U_z(t,z)\big)\vee 1=\underline{\rho}_t\vee 1=1$. Furthermore, $\bar{U}(t,0)$ is well defined. Hence, by continuity the limit of $\bar{\beta}(t, z-\mathfrak{K}_t)$ when $z - \mathfrak{K}_t \to 0$ exists and is equal to $\bar{\beta}(t, 0)$. Thus, the left-hand side of the previous equation tends to a constant $l_t < \infty$.

For the right-hand side, since by the Inada condition $\bar{U}_z(t, z-\mathfrak{K}_t) \to \infty$ when $z \to \mathfrak{K}_t$, the bracketed term can therefore only tend towards zero, that is

$$\lim_{z\to\mathfrak{K}_t}\Big[\mu_t^{\mathfrak{K}}-zr_t-C_t+P_t^{min}\rho_t^*(z)-\eta_t\cdot\delta_t^{\mathfrak{K}}+\frac{1}{2}\frac{1}{\bar{U}_{zz}\bar{U}_z}\big(||\bar{\gamma}_z^{\mathcal{R}}+\bar{U}_z\eta_t||^2-||\bar{U}_{zz}\delta_t^{\mathfrak{K},\perp}||^2\big)+\frac{\bar{\gamma}_z^{\perp}}{\bar{U}_z}\cdot\delta_t^{\mathfrak{K}}\Big]=0,$$

which we rewrite

$$\lim_{z\to\mathfrak{K}_t}\Big[\mu_t^{\mathfrak{K}}-zr_t-C_t+P_t^{min}\rho_t^*(z)-\eta_t\cdot\delta_t^{\mathfrak{K}}+\frac{\bar{\gamma}_z^{\perp}}{\bar{U}_z}\cdot\delta_t^{\mathfrak{K}}+\frac{1}{2}\Big(\frac{||\bar{\gamma}_z^{\mathcal{R}}||^2}{\bar{U}_{zz}\bar{U}_z}+2\frac{\bar{\gamma}_z^{\mathcal{R}}\cdot\eta_t}{\bar{U}_{zz}}+\frac{\bar{U}_z}{\bar{U}_{zz}}\|\eta_t\|^2-\frac{\bar{U}_{zz}}{\bar{U}_z}\|\delta_t^{\mathfrak{K},\perp}\|^2\Big)\Big]=0,$$

or equivalently,

$$\lim_{z\to\mathfrak{K}_t}\Big[A_t(z)+\frac{1}{2}\big(B_t(z)-\frac{\bar U_{zz}}{\bar U_z}\|\delta_t^{\mathfrak{K},\perp}\|^2\big)\Big]=0, \tag{3.13}$$

where we have used the notations

$$\begin{cases} A_t(z)=\mu_t^{\mathfrak{K}}-zr_t-C_t+P_t^{min}\rho_t^*(z)-\eta_t\cdot\delta_t^{\mathfrak{K}}+\dfrac{\bar\gamma_z^{\perp}}{\bar U_z}\cdot\delta_t^{\mathfrak{K}}, \\ B_t(z)=\dfrac{\|\bar\gamma_z^{\mathcal{R}}\|^2}{\bar U_{zz}\bar U_z}+2\dfrac{\bar\gamma_z^{\mathcal{R}}\cdot\eta_t}{\bar U_{zz}}+\dfrac{\bar U_z}{\bar U_{zz}}\|\eta_t\|^2. \end{cases}$$

We shall now study the limits of $A_t(z)$ and $B_t(z)$ when z tends to $\mathfrak{K}_t$. For this, we recall according to Remark 2.2, that if the Inada condition holds, then

$$\lim_{z\to\mathfrak{K}_t}\frac{\bar U_{zz}(t,z-\mathfrak{K}_t)}{\bar U_z(t,z-\mathfrak{K}_t)}=\lim_{z\to\mathfrak{K}_t}\frac{U_{zz}(t,z)}{U_z(t,z)}=+\infty.$$

Also, under Assumption 2.8, there exist a random bound ζ_t satisfying a.s. $\int_0^T\zeta_t^2dt<\infty$ for any T such that

$$\lim_{z\to\mathfrak{K}_t}\frac{\|\bar\gamma_z(t,z-\mathfrak{K}_t)\|}{\bar U_z(t,z-\mathfrak{K}_t)}\le\zeta_t$$

Thus, $|B_t(z)|=|\frac{\|\bar\gamma_z^{\mathcal{R}}\|^2}{\bar U_{zz}\bar U_z}+2\frac{\bar\gamma_z^{\mathcal{R}}\cdot\eta_t}{\bar U_{zz}}+\frac{\bar U_z}{\bar U_{zz}}\|\eta_t\|^2|\le\frac{\bar U_z}{\bar U_{zz}}(\zeta_t+3\|\eta_t\|^2)\to 0$ when $(z-\mathfrak{K}_t)\to 0$. Furthermore,

$$|A_t(z)|=|(\mu_t^{\mathfrak{K}}-zr_t-C_t+P_t^{min}\rho_t^*(z)-\eta_t\cdot\delta_t^{\mathfrak{K}})+\frac{\bar\gamma_z^{\perp}}{\bar U_z}\cdot\delta_t^{\mathfrak{K}}|\le|(\mu_t^{\mathfrak{K}}-zr_t-C_t+P_t^{min}\rho_t^*(z)-\eta_t\cdot\delta_t^{\mathfrak{K}})|+\zeta_t\|\delta_t^{\mathfrak{K},\perp}\|,$$

consequently $A_t(z)$ has a finite limit when $(z-\mathfrak{K}_t)\to 0$. This combined with $|B_t(z)|\to 0$ when $(z-\mathfrak{K}_t)\to 0$, (3.13) and the fact that $\lim_{z\to\mathfrak{K}_t}\frac{\bar U_{zz}(t,z-\mathfrak{K}_t)}{\bar U_z(t,z-\mathfrak{K}_t)}=+\infty$, implies that necessarily

$$\delta_t^{\mathfrak{K},\perp}=0. \tag{3.14}$$

It follows from (3.13) that

$$\lim_{z\to\mathfrak{K}_t}(\mu_t^{\mathfrak{K}}-zr_t-C_t+P_t^{min}\rho_t^*(z)-\eta_t\cdot\delta_t^{\mathfrak{K}})=\mu_t^{\mathfrak{K}}-\mathfrak{K}_tr_t-C_t+P_t^{min}-\eta_t\cdot\delta_t^{\mathfrak{K}}=0, \tag{3.15}$$

where we have used, according to (3.11), $\lim_{z\to\mathfrak{K}_t}\rho_t^*(z)=1$. This concludes the proof of Theorem 3.3.

3.4 Proof of Theorem 3.5

In order to show that the sustainability condition ($F_t^*\ge\mathfrak{K}_t$, $dt\otimes d\mathbb{P}$ a.s.) is satisfied under the assumptions of Theorem 3.5, we study the intermediate process $(U_z(t,F_t^*))$. This process is actually the optimal state price density process (Y_t^*). We refer to [EKHM18] for more details.

Combining the bounds (2.15) on the derivatives of the diffusion coefficient $\bar{\gamma}(t,z)$ of $\bar{U}$ and Theorem 3.3, we first show that $U_z(t,F_t^*)$ is the unique strong (non-explosive) solution of an SDE with Lipschitz coefficients. This property, combined with Inada conditions, ensures that the sustainability condition is necessarily satisfied. This preliminary result is stated in the following Proposition 3.7.

Proposition 3.7. *Under the assumptions and notations of Theorem 3.5,*

(i) The dynamics of the marginal utility U_z is given by

$$\begin{aligned} dU_z(t,z) &= \Big(-U_{zz}(t,z)\mu_t^*(z) - U_z(t,z)r_t - \frac{1}{2}U_{zzz}(t,z)\|\pi_t^*(z)\|^2 - \gamma_{zz}^{\mathcal{R}}\cdot\pi_t^*(z)\Big)dt \\ &+ \gamma_z(t,z)\cdot dW_t \end{aligned}$$

(ii) Let $\tau^F = \inf\{t \geq 0, F_t^* = \mathfrak{K}_t\}$. $U_z(t,F_t^*)$ is the solution of the following SDE on $[0,\tau^F[$ of

$$dU_z(t,F_t^*) = -U_z(t,F_t^*)r_t dt + \big(\gamma_z^{\perp}(t,F_t^*) - U_z(t,F_t^*)\eta_t\big)\cdot dW_t. \tag{3.16}$$

The proof of Theorem 3.5 is derived from Theorem 3.3 and Proposition 3.7, by following the same steps than in the proof of Theorem 4.8 in [EKHM18].

Proof of Theorem 3.5. Inspired by (3.16), let us consider the following SDE

$$dY_t = -Y_t r_t dt + \big(\gamma_z^{\perp}(t,U_z^{-1}(t,Y_t)) - Y_t\eta_t\big)\cdot dW_t.$$

By Theorem 3.3, $\delta^{\mathfrak{K},\perp} = 0$ a.s.. Hence, using the relation (2.18) between γ and $\bar{\gamma}$, the condition (2.15) can be rewritten as $||\gamma_z^{\perp}(t,z)|| \leq \zeta_t U_z(t,z)$ for some non negative process ζ such that a.s. $\int_0^T \zeta_t^2 dt < \infty$ for any T. This implies that the coefficients of this SDE are Lipschitz, yielding that this SDE admits a unique strong (non explosive) solution. Furthermore, by Proposition 3.7, $U_z(t,F_t^*)$ is solution of this SDE, which yields that $\tau^F = +\infty$ i.e. $F_t^* > \mathfrak{K}_t$, $\forall t > 0$, $\mathbb{P}$ a.s.. Besides, as a consequence of Assumptions 2.6 and 2.8, $\big(\pi_t^*(F_t^*), \rho_t^*(F^*)\big)$ verify the required integrability condition, which concludes the proof. □

The last step consists in proving Proposition 3.7.

Proof of Proposition 3.7.
(i) Since by assumption U satisfies the HJB constraint (3.5), its dynamics is

$$\begin{aligned} dU(t,z) &= \Big(-U_z(t,z)\big(zr_t + C_t - P_t^{min}\rho_t^*(z)\big) + \frac{1}{2}U_{zz}(t,z)\|\pi_t^*(z)\|^2 - V(t,\rho_t^*(z))\Big)dt \\ &+ \gamma(t,z)\cdot dW_t. \end{aligned} \tag{3.17}$$

In addition, since U is of class $\mathcal{C}^{3,\delta}$ and by Assumption 2.8, this implies by Corollary 1.3 in [EKM13], that β_z and γ_z are the local characteristics of the space derivative U_z.

On $\{1 < \rho_t^f(z)\}$, $\rho_t^*(z) = \rho_t^f(z) \in \mathcal{C}^1$ with $U_z(t,z)P_t^{min} = V_\rho(t,\rho_t^f(z))$ by (3.4). Thus $U_z(t,z)P_t^{min}\partial_z\rho_t^*(z) = V_\rho(t,\rho_t^*(z))\partial_z\rho_t^*(z)$ on $\{1 < \rho_t^f(z)\}$.
This last inequality also holds on $\{\rho_t^f(z) \leq 1\}$ since on this set, $\rho_t^*(z) \equiv 1$ implying $\partial_z\rho_t^*(z) = 0$. Using those simplifications, the derivative with respect to z of the dynamics (3.17) of U yields by [EKM13, Theorem 2.2],

$$\begin{aligned} dU_z(t,z) &= \Big(- U_{zz}(t,z)\big(zr_t + C_t - P_t^{min}\rho_t^*(z)\big) - U_z(t,z)r_t \\ &+ \frac{1}{2}U_{zzz}(t,z)\|\pi_t^*(z)\|^2 + U_{zz}(t,z)\partial_z\pi_t^*(z).\pi_t^*(z)\Big)dt + \gamma_z(t,z).dW_t. \end{aligned} \tag{3.18}$$

Now, using the definition (3.8) of π^*,

$$\partial_z\pi_t^*(z) = -\frac{U_{zzz}(t,z)}{U_{zz}(t,z)}\pi_t^*(z) - \frac{\gamma_{zz}^{\mathcal{R}}(t,F_t^*) + U_{zz}(t,F_t^*)\eta_t}{U_{zz}(t,F_t^*)}, \tag{3.19}$$

which implies that

$$\begin{aligned} &\frac{1}{2}U_{zzz}(t,z)\|\pi_t^*(z)\|^2 + U_{zz}(t,z)\partial_z\pi_t^*(z).\pi_t^*(z) \\ &= -\frac{1}{2}U_{zzz}(t,z)\|\pi_t^*(z)\|^2 - \gamma_{zz}^{R}.\pi_t^*(z) - U_{zz}(t,z)\pi_t^*(z).\eta_t. \end{aligned} \tag{3.20}$$

Using the notation $\mu_t^*(z) = zr_t + C_t - P_t^{min}\rho_t^*(z) + \pi_t^*(z)\eta_t$ for the drift of F^* and injecting (3.19) and (3.20) in the dynamics (3.18), yields

$$\begin{aligned} dU_z(t,z) &= \Big(- U_{zz}(t,z)\mu_t^*(z) - U_z(t,z)r_t - \frac{1}{2}U_{zzz}(t,z)\|\pi_t^*(z)\|^2 - \gamma_{zz}^{\mathcal{R}}.\pi_t^*(z)\Big)dt \\ &+ \gamma_z(t,z).dW_t. \end{aligned}$$

This establishes the first statement of the theorem.
(ii) Applying Itô-Ventzel's formula to $U_z(t,F_t^*)$ on $\{t < \tau^F\}$ gives (recalling the dynamics $dF_t^* = \mu_t^*(F_t^*)dt + \pi_t^*(F_t^*).dW_t$):

$$dU_z(t,F_t^*) = -U_z(t,F_t^*)r_t dt + \big(\gamma_z(t,F_t^*) + U_{zz}(t,F_t^*)\pi_t^*(F_t^*)\big).dW_t.$$

Using the identity $\gamma_z(t,F_t^*) + U_{zz}(t,F_t^*)\pi_t^*(F_t^*) = \gamma_z^{\perp}(t,F_t^*) - U_z(t,F_t^*)\eta_t$, we conclude that $U_z(t,F_t^*)$ is solution of the SDE (3.16). □

4 Application to dynamic CRRA utilities

We provide the resolution in the important example of dynamic Constant Relative Risk Aversion (CRRA) utilities, also called power dynamic utilities, introduced in 2.1. We now assume that the aggregated utility V of pensioners is given by an aggregation of individual power dynamic utilities $\bar{v}$, with pensioners having the same relative risk aversion θ.
The social planner needs to infer a dynamic utility U of the fund such that her preference

criterion (U, V) is consistent. It is then natural to search the utility U in the same class[1]. Therefore the goal is to find a consistent CRRA utility U (if it exists) satisfying assumptions of Theorem 3.5:

$$U(t,z) = Z_t^u \frac{(z - X_t)^{1-\theta}}{1-\theta}, \tag{4.1}$$

where Z^u is assumed to be a positive process with dynamics

$$dZ_t^u = Z_t^u (b_t dt + \delta_t \cdot dW_t), \quad \delta_t \in \mathbb{R}^n. \tag{4.2}$$

As discussed in Remark 3.4, to take into account the sustainability constraint, X should be taken as a buffer fund with pension rate P^{min} and contribution rate C, and that super-replicates (pathwise) the sustainability bound $\mathfrak{K}$. The dynamics of the shift X is, according to Theorem 3.3,

$$dX_t = (X_t r_t + C_t - P_t^{min})dt + \delta_t^X \cdot (dW_t + \eta_t dt), \quad \delta^X \in \mathcal{R}. \tag{4.3}$$

The dynamics of the utility process U is deduced easily from that of Z^u and X.

Lemma 4.1. *Denoting $dX_t = \mu_t^X dt + \delta_t^X.dW_t$, the dynamics of the shift X, the dynamics of the CRRA shifted utility $U(t,z) = Z_t^u \frac{(z-X_t)^{1-\theta}}{1-\theta}$ is $dU(t,z) = \beta(t,z)dt + \gamma(t,z).dW_t$, with local characteristics*

$$\begin{cases} \beta(t,z) = -U_z(t,z)(\mu_t^X + \delta_t \cdot \delta_t^X) + \frac{1}{2} U_{zz}(t,z) ||\delta_t^X||^2 + U(t,z) b_t \\ \gamma(t,z) = -U_z(t,z)\delta_t^X + U(t,z)\delta_t \end{cases}$$

If the shift X is a buffer fund (with pension rate P^{min} and contribution rate C) then $\mu_t^X = X_t r_t + C_t - P_t^{min} + \delta_t^X.\eta_t$ and $\delta^X \in \mathcal{R}$.

Proof. The proof is a straightforward application of the Itô Lemma. □

We also provide some useful relations for power utilities and their derivatives of first and second order and their conjugate:

$$\begin{cases} (z - X_t)U_z(t,z) = Z_t^u (z - X_t)^{1-\theta} = (1-\theta)U(t,z) \\ (z - X_t)^2 U_{zz}(t,z) = -\theta(z - X_t)U_z(t,z) = -\theta(1-\theta)U(t,z) \end{cases} \tag{4.4}$$

The first term $\frac{\theta}{1-\theta}(Z_t^u)^{(1/\theta)} y^{\frac{\theta-1}{\theta}}$ of the dual $\tilde{U}$ is standard, the second term $X_t y$ (linear in y) corresponds to the support function of the convex domain $[X_t, +\infty[$.

[1] If U is power then necessarily V is power with the same risk aversion coefficient, to ensure the consistency of the criterion (U, V) (see [EKHM18]).

4.1 Age-independent pensions

We start with Example 1. Each pensioner receives the same pension amount $p_t = \rho_t p_t^{min}$ (with $\rho \geq 1$), and has the same dynamic power utility $\bar{v}$,

$$\bar{v}(t,p) = Z_t \frac{(p - p_t^{min})^{1-\theta}}{1-\theta},$$

where Z is a positive adapted process. The social planner aggregates preferences of pensioners by weighting their utility by their past contributions $\alpha_c c_t(a)$, thus

$$V(t,\rho) = \omega_t^r \bar{v}(s, \rho p_t^{min}) = \omega_t^r Z_t \, (p_t^{min})^{1-\theta} \, \frac{(\rho-1)^{1-\theta}}{1-\theta} \tag{4.5}$$

where $\omega_t^r = \alpha_c \int_{a_r}^{\infty} c_t(a) n(t,a) da$ is the total weight of all pensioners living at time t.

4.1.1 Optimal policy

Proposition 4.2. *Assume that the shifted power utility U defined by (4.1) is a consistent utility verifying assumptions of Theorem 3.5. Then the optimal strategy is given by*

$$\begin{cases} \pi_t^* = \delta_t^X + \dfrac{1}{\theta}(F_t^* - X_t)(\delta_t^{\mathcal{R}} + \eta_t) & (4.6) \\ p_t^* = p_t^{min} + (F_t^* - X_t)\left(\dfrac{Z_t \omega_t^r}{Z_t^u N_t^r}\right)^{\frac{1}{\theta}} & (4.7) \end{cases}$$

and the optimal fund F^* is solution of the dynamics

$$dF_t^* = dX_t + (F_t^* - X_t)\left(\left(r_t + (\frac{Z_t \omega_t^r}{Z_t^u N_t^r})^{\frac{1}{\theta}}\right) dt + \frac{1}{\theta}(\delta_t^{\mathcal{R}} + \eta_t)\cdot\big(dW_t + \eta_t dt\big)\right). \tag{4.8}$$

Note that due to the form of the pension utility $\bar{v}$, ρ_t^f defined by (3.4) is always greater than the bound 1 and thus $\rho_t^* = \rho_t^f$ a.s.

The optimal strategy has a particular additive form. The optimal pension is the minimal pension p_t^{min} plus an additional term that is proportional to the "cushion" $(F_t^* - X_t)$. The proportionality factor $\left(\frac{Z_t \omega_t^r}{Z_t^u N_t^r}\right)^{\frac{1}{\theta}}$ is decreasing in the risk aversion parameter θ. The ratio $\frac{\omega_t^r}{N_t^r} = \frac{\int_{a_r}^{\infty} \alpha_c c_t(a) n(t,a) da}{\int_{a_r}^{\infty} n(t,a) da}$ represents the average past contributions of one pensioner at time t. Thus, giving the same pension to all pensioners yields a risk sharing among pensioners. The ratio $\frac{Z_t}{Z_t^u}$ represents the relative importance of the pensioners utility with respect to the buffer fund utility.

Similarly, the optimal portfolio is the portfolio δ^X of buffer fund shift X plus an additional term proportional to the "cushion" $(F_t^* - X_t)$. Again the proportionality factor $\frac{(\delta_t^{\mathcal{R}} + \eta_t)}{\theta}$ is decreasing in the risk aversion parameter θ, but does not depend on demographic components.

Proof. By a simple derivation with respect to z and ρ, we have $U_z(t,z) = Z_t^u (z - X_t)^{-\theta}$

and $V_\rho(t,\rho) = Z_t\omega^r_s p^{min}_t (p^{min}_t(\rho-1))^{-\theta}$ so that $p^{min}_t V^{-1}_\rho(t,y) = p^{min}_t + \left(\frac{y}{Z_t\omega^r_s p^{min}_t}\right)^{-\frac{1}{\theta}}$. Thus, applying Theorem 3.5

$$\begin{aligned}
p^*_t &= p^{min}_t V^{-1}_\rho(t, P^{min}_t U_z(t,F^*_t)) \\
&= p^{min}_t V^{-1}_\rho(t, P^{min}_t Z^u_t (F^*_t - X_t)^{-\theta}) \\
&= p^{min}_t + \left(\frac{Z^u_t (F^*_t - X_t)^{-\theta} P^{min}_t}{Z_t\omega^r_t p^{min}_t}\right)^{-\frac{1}{\theta}} \\
&= p^{min}_t + \left(\frac{Z_t\omega^r_t}{Z^u_t N^r_t}\right)^{\frac{1}{\theta}} (F^*_t - X_t).
\end{aligned}$$

where we used $P^{min}_t = p^{min}_t N^r_t$ for the last equality. In addition, the optimal investment strategy is given by

$$\begin{aligned}
\pi^*_t &= -\frac{\gamma^{\mathcal{R}}_x(t,F^*_t) + U_z(t,F^*_t)\eta_t}{U_{zz}(t,F^*_t)} \\
&= -\frac{-U_{zz}(t,F^*_t)\delta^X_t + U_z(t,F^*_t)(\delta^{\mathcal{R}}_t + \eta_t)}{U_{zz}(t,F^*_t)} \\
&= \delta^X_t + \frac{1}{\theta}(F^*_t - X_t)(\delta^{\mathcal{R}}_t + \eta_t)
\end{aligned}$$

where we used Lemma 4.1. Plugging this strategy into the buffer fund dynamics (1.14) yields

$$\begin{aligned}
dF^*_t \quad &= \quad \left(F^*_t r_t + C_t - N^r_t p^{min}_t \left(1 + \left(\frac{Z_t\omega^r_t}{Z^u_t N^r_t}\right)^{\frac{1}{\theta}} (F^*_t - X_t)\right)\right) dt \\
&+ \quad \left(\frac{1}{\theta}(F^*_t - X_t)(\delta^{\mathcal{R}}_t + \eta_t) + \delta^X_t\right).(dW_t + \eta_t dt).
\end{aligned}$$

which implies (4.8) by (4.3). $\square$

4.1.2 Consistency

The next step consists in verifying the assumptions of Theorem 3.5 and in particular the consistency condition, that translates into condition on Z^u.

Proposition 4.3. *Let V be the aggregated pensioners utility (4.5) and $U(t,z) = Z^u_t \frac{(z-X_t)^{1-\theta}}{1-\theta}$. If (U,V) is consistent then Z^u must be a well-defined solution to the SDE:*

$$-dZ^u_t = Z^u_t \left(\left((1-\theta)r_t + \frac{(1-\theta)}{2\theta}||\delta^{\mathcal{R}}_t + \eta_t||^2 + \theta(Z_t\omega^r_t)^{\frac{1}{\theta}}(N^r_t)^{\frac{\theta-1}{\theta}}(Z^u_t)^{\frac{-1}{\theta}}\right) dt - \delta_t.dW_t\right). \quad (4.9)$$

Proof. One the one hand, by application of Lemma 4.1, the buffer fund dynamic utility U

has the following dynamics

$$\begin{cases} dU(t,z) = \beta(t,z)dt + \gamma(t,z)dW_t, \\ \beta(t,z) = -U_z(t,z)\big(X_t r_t + C_t - P_t^{min} + \delta_t^X \cdot (\eta_t + \delta_t)\big) + \frac{1}{2}U_{zz}(t,z)||\delta_t^X||^2 + U(t,z)b_t \\ \gamma(t,z) = -U_z(t,z)\delta_t^X + U(t,z)\delta_t. \end{cases} \tag{4.10}$$

On the other hand, if (U,V) is consistent the consistency constraint (3.5) should be satisfied:

$$\beta(t,z) = -U_z(t,z)\big(zr_t + C_t - P_t^{min}\rho_t^*(z)\big) + \frac{1}{2}U_{zz}(t,z)\|\pi_t^*(z)\|^2 - V(t,\rho_t^*(z)).$$

By the expression of optimal controls given in Proposition 4.2, the elementary identities (4.4) and using (4.5) and (4.6) we can rewrite the consistency constraint (3.5) as

$$\beta(t,z) = -U_z(t,z)\big(zr_t + C_t - P_t^{min} + \delta_t^X \cdot (\eta_t + \delta_t)\big) + \frac{1}{2}U_{zz}(t,z)||\delta_t^X||^2 \\ + U(t,z)\left((1-\theta)N_t^r\big(\frac{Z_t\omega_t^r}{Z_t^u N_t^r}\big)^{\frac{1}{\theta}} - \frac{(1-\theta)}{2\theta}\|\delta^{\mathcal{R}} + \eta_t\|^2 - \omega_t^r \frac{Z_t}{Z_t^u}\big(\frac{Z_t\omega_t^r}{Z_t^u N_t^r}\big)^{\frac{1}{\theta}-1}\right).$$

Identifying this with (4.10), we obtain

$$-U_z(t,z)X_t r_t + U(t,z)b_t = -U_z(t,z)zr_t \\ +U(t,z)\left((1-\theta)N_t^r\big(\tfrac{Z_t\omega_t^r}{Z_t^u N_t^r}\big)^{\frac{1}{\theta}} - \tfrac{(1-\theta)}{2\theta}\|\delta^{\mathcal{R}} + \eta_t\|^2 - \omega_t^r \tfrac{Z_t}{Z_t^u}\big(\tfrac{Z_t\omega_t^r}{Z_t^u N_t^r}\big)^{\frac{1}{\theta}-1}\right)$$

which implies that the drift b_t of Z^u must satisfy

$$b_t = -(1-\theta)r_t - \frac{(1-\theta)}{2\theta}\|\delta^{\mathcal{R}} + \eta_t\|^2 - \theta(N_t^r)^{1-\frac{1}{\theta}}\big(\frac{\omega_t^r Z_t}{Z_t^u}\big)^{\frac{1}{\theta}}.$$

□

SDE (4.9) is only defined on $Z^u > 0$, due to the term $(Z_t^u)^{-\frac{1}{\theta}}$ in the drift.

Proposition 4.4. *Let ξ be the process defined by*

$$\xi_t = exp\left(\int_0^t \big((1-\theta)r_t + \frac{(1-\theta)}{2\theta}||\delta_t^{\mathcal{R}} + \eta_t||^2 + \frac{\|\delta_s\|^2}{2}\big)ds - \int_0^t \delta_s \cdot dW_s\right)$$

There exists a unique solution of the SDE (4.9), defined by

$$Z_t^u = \xi_t^{-1}\left(Z_0^{\frac{1}{\theta}} - \int_0^t (N_s^r)^{1-\frac{1}{\theta}}(Z_s\omega_s^r\xi_s)^{\frac{1}{\theta}}ds\right)^{\theta}, \quad \forall t \in [0,\tau^Z[, \tag{4.11}$$

where τ^Z the first hitting time of 0 satisfies

$$\tau^Z = \inf\{t\ ;\ \int_0^t (N_s^r)^{1-\frac{1}{\theta}}\big(\frac{Z_s}{Z_0}\omega_s^r\xi_s\big)^{\frac{1}{\theta}}ds \geq 1\}. \tag{4.12}$$

Proof. Let Z^u be a solution of (4.9) on $[0,\tau[$, with $\tau = \inf\{t \geq 0;\ Z^u_t = 0\}$, and denote

$$\tilde{r}_t := (1-\theta)r_t + \frac{(1-\theta)}{2\theta}||\delta^{\mathcal{R}}_t + \eta_t||^2, \qquad \text{and } \tilde{\beta}_t := (N^r_t)^{1-\frac{1}{\theta}}(Z_t\omega^r_t)^{\frac{1}{\theta}},$$

so that

$$dZ^u_t = -\big(Z^u_t\tilde{r}_t + \theta\tilde{\beta}_t(Z^u_t)^{1-\frac{1}{\theta}}\big)dt + Z^u_t\delta_t \cdot dW_t.$$

This equation may be simplified by the following change of variables:

$$\bar{Z}_t := Z^u_t \exp\left(\int_0^t (\tilde{r}_s + \frac{\|\delta_s\|^2}{2})ds - \int_0^t \delta_s \cdot dW_s\right) = Z^u_t\xi_t$$

Then,

$$\frac{1}{\theta}(\bar{Z}_t)^{\frac{1}{\theta}-1}d\bar{Z}_t = -\beta_t\xi_t^{\frac{1}{\theta}}dt,$$

i.e

$$(\bar{Z}_t)^{\frac{1}{\theta}} - (\bar{Z}_0)^{\frac{1}{\theta}} = -\int_0^t \beta_s\xi_s^{\frac{1}{\theta}}ds,$$

and thus

$$Z^u_t = \xi_t^{-1}\left(Z_0^{\frac{1}{\theta}} - \int_0^t (N^r_s)^{1-\frac{1}{\theta}}(Z_s\omega^r_s\xi_s)^{\frac{1}{\theta}}ds\right)^\theta,$$

and τ^Z verifies (4.12). □

The necessary condition for Z^u given in Proposition 4.3 is actually a sufficient condition.

4.1.3 Existence of the optimal investment/pension policy

Theorem 4.5. *Let Z^u be defined as in Proposition 4.4. Then the shifted power dynamic utilities (U,V) given by (4.1)-(4.5) are consistent on $[0,\tau^Z[$, and verify the assumption of Theorem 3.5 on this interval. Therefore the optimal strategy is given on $[0,\tau^Z[$ by*

$$\begin{cases} \pi^*_t = \delta^X_t + \frac{1}{\theta}(F^*_t - X_t)(\delta^{\mathcal{R}}_t + \eta_t) \\ p^*_t = p^{min}_t + (F^*_t - X_t)\left(\frac{Z_t\omega^r_t}{Z^u_tN^r_t}\right)^{\frac{1}{\theta}}. \end{cases}$$

Furthermore, $\tau^Z = \inf\{t \geq 0;\ F^*_t = X_t\}$.

Observe that p^* depends on the population dynamics only through the ratio ω^r_t/N^r_t. Since all pensioners receive the same pension amount p^*_t, there is also an intergenerational risk-sharing among pensioners.

For $t \geq \tau^Z$, $Z^u_t = 0$ and thus the buffer fund utility $U(t,\cdot) \equiv 0$. At this stopping time, the buffer fund hits the boundary X. In order to stay sustainable, the pension system parameters have to be updated, for instance by decreasing the sustainability bound $\mathfrak{K}_t$, the minimum pension amount $p^{min}_t(a)$ or the importance given to the pensioners' preferences. Observe that thanks to (4.12), the distribution τ^Z can be approximated numerically. In particular, τ^Z decreases with the number of pensioners and the weight of pensioner's preferences attributed by the social planner. However, τ^Z does not depend on the initial buffer fund amount F_0, due to the choice of dynamic power utilities.

Proof of Theorem 4.5. The consistency SDE is verified by Proposition 4.3 and 4.4. It is straightforward to check Assumption 2.8, since $\bar{U}(t,z) = U(t, z - X_t) = \frac{z^{1-\theta}}{1-\theta}$.
Furthermore, $\tilde{V}(t,z) = z + \frac{\theta}{1-\theta}(\omega_t^r Z_t (p_t^{min})^{1-\theta})^{\frac{1}{\theta}} z^{1-\frac{1}{\theta}} = z + \frac{\theta}{1-\theta}(\omega_t^r Z_t)^{\frac{1}{\theta}} \left(\frac{z}{p_t^{min}}\right)^{1-\frac{1}{\theta}}$, and it is straightforward to check that Assumption 2.6 is verified. The optimal strategy is then obtained from Proposition 4.2. By Proposition 4.2, and using the short notation $\mathcal{E}(\int_0^t g_s \cdot dW_s)$ for the Doléans-Dade exponential martingale[2]

$$(F_t^* - X_t) = \big(F_0^* - X_0\big)\exp\Big(\int_0^t (r_s - (N_s^r)^{1-\frac{1}{\theta}} (\frac{Z_s\omega_s^r}{Z_s^u})^{\frac{1}{\theta}} + \frac{1}{\theta}(\delta_s^{\mathcal{R}} + \eta_s)\cdot\eta_s)ds\Big)\mathcal{E}\Big(\int_0^t \frac{1}{\theta}(\delta_s^{\mathcal{R}} + \eta_s)\cdot dW_s\Big),$$

and by Proposition 4.3,

$$Z_t^u = Z_0^u \exp\Big(-\int_0^t \Big(((1-\theta)r_s + \frac{(1-\theta)}{2\theta}||\delta_s^{\mathcal{R}} + \eta_s||^2 + \theta(N_s^r)^{1-\frac{1}{\theta}} (\frac{Z_s\omega_s^r}{Z_s^u})^{\frac{1}{\theta}} ds\Big)\mathcal{E}\Big(\int_0^t \delta_s \cdot dW_s\Big).$$

Then,

$$(F_t^* - X_t)(Z_t^u)^{-\frac{1}{\theta}} = \big(F_0^* - X_0\big)(Z_0^u)^{-\frac{1}{\theta}} \exp\Big(\frac{1}{\theta}\int_0^t (r_s + \frac{1}{2}||\delta_s^{\perp} - \eta_s||^2)ds - \int_0^t \frac{1}{\theta}(\delta_s^{\perp} - \eta_s)\cdot dW_s\Big).$$

The previous equation can be rewritten as

$$(F_t^* - X_t)(Z_t^u)^{\frac{-1}{\theta}} = \big(F_0^* - X_0\big)(Z_0^u)^{\frac{-1}{\theta}}(Y_t)^{\frac{-1}{\theta}},$$

where we recognize the state price density process Y

$$Y_t = \exp\left(-\int_0^t r_s ds\right)\mathcal{E}\Big(\int_0^t (\delta_s^{\perp} - \eta_s)\cdot dW_s\Big).$$

In particular $\tau^Z = \inf\{t \geq 0;\ F_t^* = X_t\}$. □

Section 4.2 explains how to extend the results obtained for Example 1 to Example 2.

4.2 Age-dependent pensions

In Example 2 of age-dependent pension and utility $\bar{v}$, the aggregate utility of pension V is a complex aggregation between cohorts given by (2.8)

$$V(t,\rho) = \int_{a_r}^{\infty} \bar{v}(t, a, \rho\, p_{ret}(a_r + t - a)e^{\int_{a_r+t-a}^t \lambda_u du})\, n(t,a)da.$$

[2] $\mathcal{E}(\int_0^t g_s \cdot dW_s) := \exp\left(\int_0^t g_s \cdot dW_s - \frac{1}{2}\int_0^t ||g_s||^2 ds\right)$.

We assume that pensioner utility depends on the age a through the shift $p^{min}(t,a)$ in the shifted power utility

$$\bar{v}(t,a,p) = Z_t \frac{(p - p_t^{min}(a))^{1-\theta}}{1-\theta} \quad \text{and} \quad v(t,a,\rho) = (p_t^{min}(a))^{1-\theta} Z_t \frac{(\rho-1)^{1-\theta}}{1-\theta}.$$

Therefore in the case of power utility, the aggregate utility of pension has the following multiplicative form

$$V(t,\rho) = \tilde{\omega}_t^r Z_t \frac{(\rho-1)^{1-\theta}}{1-\theta}, \qquad \text{with} \quad \tilde{\omega}_t^r = \int_{a_r}^{\infty} (p_t^{min}(a))^{1-\theta} n(t,a) da.$$

Observe that the problem is formulated similarly than in Example 1, with a different weight $\tilde{\omega}_t^r$ and in this case $P_t^{min} = \int_{a_r}^{\infty} p^{min}(t,a) n(t,a) da$. Thus similar computations as in Section 4.1 yield

$$\begin{aligned}
\rho_t^* &= V_\rho^{-1}(t, P_t^{min} U_z(t, F_t^*)) \\
&= V_\rho^{-1}(t, P_t^{min} Z_t^u (F_t^* - X_t)^{-\theta}) \\
&= 1 + \left(\frac{Z_t^u (F_t^* - X_t)^{-\theta} P_t^{min}}{Z_t \tilde{\omega}_t^r} \right)^{-\frac{1}{\theta}} \\
&= 1 + (F_t^* - X_t) \left(\frac{Z_t}{Z_t^u} \frac{\int_{a_r}^{\infty} (p_t^{min}(y))^{1-\theta} n(t,y) dy}{\int_{a_r}^{\infty} p_t^{min}(y) n(t,y) dy} \right)^{\frac{1}{\theta}}.
\end{aligned}$$

Therefore the optimal strategy in this Example 2 with CRRA utility is (on $[0, \tau^Z[$)

$$\begin{cases}
\pi_t^* = \delta_t^X + \dfrac{1}{\theta}(F_t^* - X_t)(\delta_t^{\mathcal{R}} + \eta_t) \\
p_t^*(a) = p_t^{min}(a) \Big(1 + (F_t^* - X_t) \left(\dfrac{Z_t}{Z_t^u} \dfrac{\int_{a_r}^{\infty} (p_t^{min}(y))^{1-\theta} n(t,y) dy}{\int_{a_r}^{\infty} p_t^{min}(y) n(t,y) dy} \right)^{\frac{1}{\theta}} \Big).
\end{cases}$$

Observe that in Examples 1 and 2, the portfolio are the same. What differs are the pensions. Nevertheless if $p_t^{min}(a) = p_t^{min}$ does not depend on the age, then the optimal pension $p_t^*(a)$ simplifies in $p_t^*(a) = p_t^* = p_t^{min} + (F_t^* - X_t) \left(\frac{Z_t}{Z_t^u} \right)^{\frac{1}{\theta}}$. It is coherent with Example 1 since it corresponds to the particular case of Example 1 in which each pensioner has weight 1, that is $\omega_t^r = N_t^r$.

The increase of the pension amount with age depends on the indexation rate λ_t in $p_t^{min}(a)$ and on the optimal adjustment $\rho_t^* = 1 + (F_t^* - X_t) \left(\frac{Z_t}{Z_t^u} \frac{\int_{a_r}^{\infty} (p_t^{min}(y))^{1-\theta} n(t,y) dy}{\int_{a_r}^{\infty} p_t^{min}(y) n(t,y) dy} \right)^{\frac{1}{\theta}}$. As in Example 1, ρ^* still depends on the importance attributed to the pensioners' preferences with respect to Z^u. The pension can also be written as follows, to make appear the relative weight of cohort

a, captured through $p_t^{min}(a)$, with respect to the other cohorts:

$$p_t^*(a) = p_t^{min}(a) + (F_t^* - X_t)\left(\frac{Z_t}{Z_t^u}\frac{\int_{a_r}^{\infty}(\frac{p_t^{min}(y)}{p_t^{min}(a)})^{1-\theta}n(t,y)dy}{\int_{a_r}^{\infty}(\frac{p_t^{min}(y)}{p_t^{min}(a)})n(t,y)dy}\right)^{\frac{1}{\theta}}.$$

Conclusion This paper designs a social planner's dynamic decisions criterion under sustainability, adequacy and fairness constraints. The optimal investment/pension policy is derived when the social planner can invest in/borrow from a buffer fund, with the aim to provide a better demographic and financial risk-sharing across generations. This flexible modeling can be easily extended to heterogenous cohorts or open populations, thus considering also intra generational risk sharing.
The explicit computations for the optimal policies allows these theoretical results to be applied to an empirical setting with real data. For instance, an interesting application will consist in computing and analyzing actuarial fairness criteria for each generation, and to test the impact of demographic shocks (such as the "baby boom") on optimal contribution/benefit plans.

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