Estimating the conditional Expected-Shortfall parameter

Christian Francq and Jean-Michel Zakoïan

CREST, 5 Avenue Henry Le Chatelier, 91120 Palaiseau

Keywords: Estimation Risk, Quantile Regression, Risk measures, Value-at-Risk, Volatility models.

1 Introduction

The Value-at-Risk (VaR) and Expected Shorfall (ES) have become predominant measures of market risk of financial assets. While VaR is simply defined as the opposite of an α-quantile of the loss distribution, ES is the expected loss given the loss exceeds the VaR at level α. These risk measures have been initially defined as fixed parameters of the marginal distribution of the returns (possibly computed on rolling windows). Financial crises periodically lead to question the definition of risk measures as time-constant quantities. Modern financial risk management thus considers risk measures based on conditional distributional information, and views risk as a stochastic process.

Conditional risk measures allow to adjust the reserves dynamically, depending on the current information. From a statistical point of view, evaluating conditional risks poses serious challenges as (i) the underlying dynamic of the loss has to be estimated in a preliminary step, and (ii) the estimation of the parameters is likely to impact the accuracy of the estimated risk measures.

Despite ES, unlike VaR, is known to be a coherent measure of risk (Artzner, et al. 1999), and the Basel III Accord places particular attention on ES, there is little existing work on modeling conditional ES. Linton and Xiao (2013) considered nonparametric estimation of the conditional quantiles and ES. Patton, Ziegel and Chen (2017) propose and study new generalized autoregressive score (GAS) models for conditional ES and VaR. We propose a semi-parametric approach.

2 Semi-parametric framework

In this project we consider a general *semiparametric* specification $\epsilon_t = \sigma_t \eta_t$ for an observed sequence of returns ϵ_t. The volatility σ_t is assumed to follow a GARCH type equation of the form $\sigma_t = \sigma(\epsilon_{t-1}, \epsilon_{t-2}, \dots; \boldsymbol{\theta}_0)$ where $\boldsymbol{\theta}_0$ is a vector-valued unknown parameter. In this semi-parametric framework, the distribution of the independent and identically distributed (iid) innovations η_t is not specified. Being agnostic about the distribution of η_t is particularly relevant since the tail-thickness and skewness of the conditional distribution of the returns cannot be easily represented by parametric models. For identifiability reasons and interpretability, the innovations are assumed to have zero-mean and unit variance.

The *conditional* VaR of the process (ϵ_t) at risk level $\alpha \in (0, 1)$, denoted by $\mathrm{VaR}_t(\alpha)$, is defined as the opposite of the α-quantile of the conditional distribution of ϵ_t, while the *conditional* ES at the same risk level, denoted by $\mathrm{ES}_t(\alpha)$, is defined as the conditional expected loss given that the loss is larger than the VaR. An attractive property of the model is that both conditional risk measures have a multiplicative form. The conditional VaR is the product of the volatility and the opposite of a quantile ξ_α of the innovations, that is the marginal VaR of η_t. The conditional ES shares the same property, but the marginal VaR is replaced by the marginal ES μ_α of the innovations.

In practice, the conditional risk measures have to be estimated from observations of the returns. The advantage of the multiplicative form is that the estimation can be conducted in two steps: i) estimation of the volatility through the parameter $\boldsymbol{\theta}_0$; estimation of the characteristics (ξ_α or μ_α) of the innovations distribution. For step i), a convenient and widely used method is the Gaussian Quasi-Maximum Likelihood (QML) method which does not require postulating a specific distribution for the innovations. Concerning Step ii), of course the innovations are not observed but they can be replaced by the residuals obtained from the first step. The characteristics of the errors distribution can then be estimated using straightforward sample quantiles and moments.

3 Key issues and findings

Two key issues considered in this project were the following:

- Can we derive Confidence Intervals (CI) for the conditional VaR and ES?
- Can we derive tests for the characteristics of the innovations distribution?

Concerning the first issue, we derived the joint asymptotic distribution of the volatility parameter estimator and the empirical characteristic of the residuals (Francq and Zakoïan (2015, 2022a)). CI for the VaR and ES based on the asymptotic distribution can be derived as a consequence. Drawing such CIs allows to take into account the *estimation risk* inherent to the evaluation of the VaR and ES, and to visualize simultaneously the estimation and market risks. For risk managers, such IC's provide indications on how to modify the reserves, in order to account for the uncertainty in the evaluation of the model parameters.

Concerning the second issue, we tested several simplifying assumptions about the gaussianity or the symmetry of the innovations quantiles. Under these hypotheses, risk management can be simplified and the statistical procedures can be more efficient. We derived the distributions of the tests in Francq and Zakoïan (2022b) and illustrated them by numerous numerical examples on real and simulated data. One important finding is that such tests depend crucially on the estimation: neglecting it (i.e. applying tests designed for i.i.d. data) would lead to an uncontrolled proportion of wrong decisions. More surprisingly, the tests do not depend on the volatility model: they are thus simple to implement.

The practical implication of this work will be to offer new tools for evaluating the estimation uncertainty related to the evaluation of the dynamics of the market risk.

References

Artzner, P., Delbaen, F., Eber, J.M. and D. Heath (1999) Coherent measures of risk *Mathematical Finance* 9, 203–228.

Francq, C. and J-M. Zakoïan (2015) Risk parameter estimation in volatility models. *Journal of Econometrics* 184, 158–173.

Francq, C. and J-M. Zakoïan (2022a) Adaptiveness of the empirical distribution of residuals in semi-parametric conditional location scale models. *Bernoulli* 28, 548–578.

Francq, C. and J-M. Zakoïan (2022b) Testing hypotheses on the innovations distribution in semi-parametric conditional volatility models. In revision for *Journal of Financial Econometrics.*

Linton, O., and Z. Xiao (2013) Estimation of and inference about the Expected Shortfall for time series xith infinite variance. *Econometric Theory* 29, 771–807.

Patton, A.J., Ziegel, J.F. and R. Chen (2017) Dynamic Semiparametric Models for Expected Shortfall (and Value-at-Risk). *Preprint* `http://public.econ.duke.edu/~ap172/`