Economic Modelling of the Bitcoin Mining Industry

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joint work with
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15th Financial Risks International Forum

March 22, 2022

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1 Introduction

Motivations

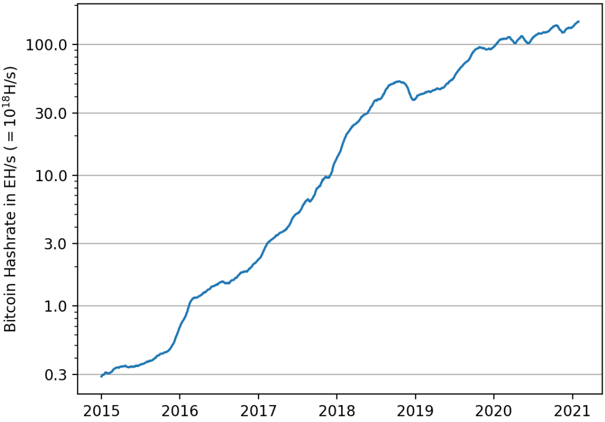


Figure: Total Bitcoin hashrate - log scale

Motivations

- On Proof-of-Work blockchains, the total hashrate is an important determinant of the consensus protocol
- We build a homogenous framework to understand the hashrate dynamics
- The hashrate is the result of miners interactions through computational power
- To understand the hashrate we need to understand the market for computational power

Strategy of the Paper

1. Derive a homogenous framework
2. Numerically solve for equilibrium
3. Use the numerical solution to calibrate parameters based on publicly observable Bitcoin data
4. Deduce consequences on :
   - **Energy consumption**
   - Security against 51% attacks
   - R&D investment of mining machine manufacturers

Literature review

- Cryptocurrency valuation : Biais, Bisiere, Bouvard, Casamatta and Menkveld (2020), Schilling and Uhlig (2019), Pagnotta (2020)
- Mining pools : Schrijvers, Bonneau, Boneh and Roughgarden (2016), Fisch, Pass and Shelat (2017), Cong, He and Li (2019)
- Mining concentration : Alsabah and Capponi (2019), Li, Reppen and Sircar (2019)
- Equilibrium analysis of mining : Prat and Walter (2021)
- More general framework : Bertucci, Bertucci, Lasry and Lions (2020)

2 An Homogenous Model of Bitcoin Mining

Model Setup

Model assumptions

- Continuous-time environment ; time is discounted at rate r
- Perfect mining diversification ; constant reward in cryptocurrency
- Miners revenue (block reward + fees) is R given by

$$\frac{dR}{R} = \alpha dt + \sigma dW_t$$

Technological progress - Real Hashrate

- assume constant technological progress rate : δ
- "real hashrate", K_t, is the "nominal hashrate", P_t, discounted by δ

$$K_t := e^{-\delta t} P_t$$

$\Longrightarrow$ focus on the value of 1 unit of real hashrate, denoted by u

Market for Mining Devices

- Miners value a machine of the latest generation at : $p = u$
- Assume the quantity available at any time is given by

$$Q(p) = \lambda K(p - \overline{p})$$

$\Longrightarrow$ Important homogeneity assumption here

- Therefore in equilibrium we have that

$$Q^* = Q(u) = \lambda K(u - \overline{p})$$

Real Hashrate Dynamics

$$\frac{\dot{K}}{K} = -\delta + \lambda(u - \overline{p})$$

- The real hashrate depreciates at the rate of technological progress
- Miners decisions are continuous
- From the equilibrium of the mining hardware industry we know miners' contribution to the growth rate of the real hashrate is $\lambda(u - \overline{p})$

Value of one unit of real hashrate

- Denote by c the electricity cost associated to running one unit of real hashrate
- Value function :

$$u(K,R) := \int_0^\infty e^{-(r+\delta)t}\left(\frac{R_t}{K_t} - c\right) dt$$

where $(K_t)_{0\leq t}$ and $(R_t)_{t\geq 0}$ are processes satisfying

$$\begin{cases} \frac{\dot{K}_t}{K_t} = -\delta + \lambda(u(K_t,R_t) - \overline{p}) \\ dR_t = \alpha R_t dt + \sigma R_t dW_t \\ K_0 = K; R_0 = R \end{cases}$$

Equilibrium equation

- No-arbitrage equilibrium equation

$$u(K_t, R_t) = \mathbb{E}\left[e^{-(r+\delta)t}u(K_{t+dt}, R_{t+dt}) + \left(\frac{R_t}{K_t} - c\right)dt\right]$$

- which can be rewritten as

$$0 = -(r-\delta)U + K\partial_K U(-\delta + \lambda(U - \overline{p})) + \alpha R\partial_R U + \frac{\sigma^2}{2}R^2\partial_{RR}U + \frac{R}{K} - c$$

Scale invariance - Benevolent Planer

- Because of our homogeneity assumptions we can rewrite the problem in terms of the revenue per machine ($z = ln(\frac{R}{K})$)
- After a few math steps, we obtain the social planer equation

$$0 = -(r+\delta)V(z) + V'(z)\left(\delta + \alpha - \frac{\sigma^2}{2}\right) + \frac{\sigma^2}{2}V''(z) - \frac{\lambda}{2}\left(V'(z) - \overline{p}\right)^2 + e^z - cz$$

3 Model Calibration on Data

Taking the model to the data

- We solve the HJB equation using standard numerical scheme (Godunov/Newton) with Neumann boundary conditions
- We use a structural approach to calibrate the mining hardware manufacturers supply function

$$Q(U) = \lambda K(U - \overline{p})$$

Data

- We use publicly available data of the Bitcoin Blockchain
- Total miners' revenue : $\{R_t\}_t$
- Nominal Hashrate : $\{P_t\}_t$
- Between January 1st, 2015 and February 1st, 2021

Parameters' Calibration

Exogenous parameters

Description	Parameter	Value
Discount Rate	r	0.2
Drift in reward process	α	0.652
Diffusion in reward process	σ	0.679
Technological rate of progress	δ	0.42
Unitary cost of electricity	c	350
Lag of hashrate growth	τ	3 months
Real hashrate normalization	-	29.23

Endogenous Parameters

- Real hashrate Dynamics is $\frac{\dot{K}}{K} = -\delta + \lambda(U - \overline{p})$
- We minimize the distance

$$(\lambda^*, \overline{p}^*) = \min_{\lambda, \overline{p}} \Gamma(\lambda, \overline{p}) = \sum_t \left(\frac{K_{t+\tau} - K_t}{K_t} - \left(-\delta + \lambda \left(U_{\lambda, \overline{p}} \left(\frac{R_t}{K_t} \right) - \overline{p} \right) \right) \right)^2$$

Main Results (1)

- $\lambda = 7 \times 10^{-4}$ and $\overline{p} = 775$

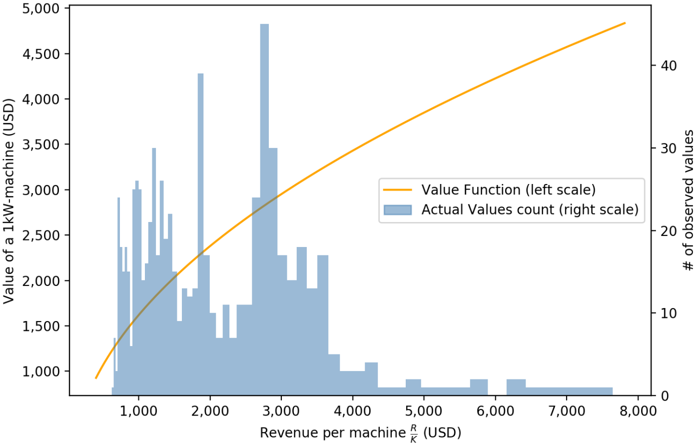


Figure: Value function (orange - left) and values for $x = \frac{R}{K}$ (blue - right)

Main Results (2)

- correlation = 63% (***)

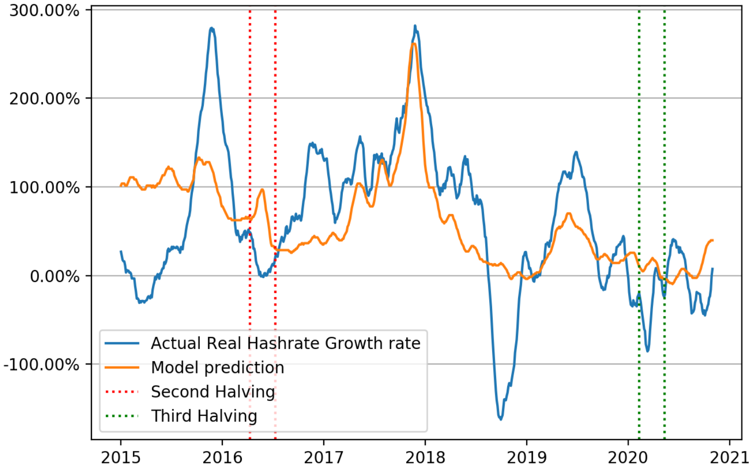


Figure: Actual (blue) and prediction (orange) of real hashrate growth rate

Revenue per machine

- Model assumes : both miners reward and real hashrate grow at the same constant rate on average

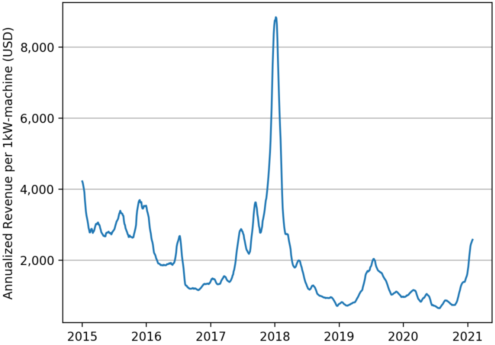


Figure: Revenue per machine that consumes 1kW of energy at current efficiency

- Law of motion of z :

$$dz_t = (\delta - \lambda(\tilde{v}(z) - \overline{p}) + \alpha - \nu)\, dt + \sigma dW_t$$

Equilibrium responses to shocks on the bitcoin price

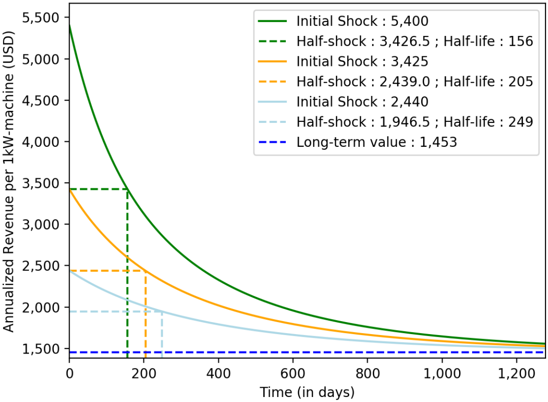


Figure: Dynamics of the revenue per machine after a shock on the Bitcoin price (without noise)

Cost analysis of Bitcoin miners

Bitcoin/USD Price after the shock	Hardware investment (for first year in USD b)	Electricity spendings (for first year in USD b)	Total Revenue (for first year in USD b)
40,000 (no shock)	21.17	4.46	18.51
60,000	35.4	5.14	27.77
80,000	50.05	5.77	37.02
100,000	65.01	6.34	46.28
120,000	80.13	6.9	55.54

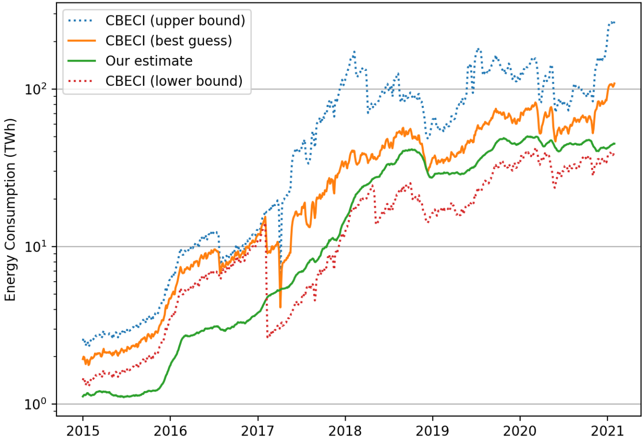
Table: Mining costs for one year following an increase in the bitcoin price with an initial bitcoin/USD price of USD 40,000

- Electricity's price drives miners' profitability
- But the biggest cost is by far the hardware investment needed to stay competitive

4 Energy Consumption of Bitcoin Mining

Total Bitcoin mining energy consumption

- Energy consumption is directly proportional to the real hashrate K
- And we know that on the long run, K grows with miners reward, as does therefore energy consumption



Discussion around energy consumption

- The energy consumption increased exponentially because the bitcoin price did...
- ... but it will likely stabilize at some point.

- This is only the energy consumption and a deeper analysis would be needed to understand how much carbon this will emit :
  - Not all energy sources have the same characteristics
  - They don't have the same comparative advantage to be used for bitcoin mining...

Conclusion

- We have built a homogenous model for analyzing the Bitcoin production industry
- Our parsimonious model is able to fit well the data
- The mining equilibrium is quite stable, which indicates resiliency and security
- Implications of the model :
  - Energy consumption grows linearly with the mining reward
  - Blockchain security grows linearly with the mining reward
  - R&D spendings grow linearly with the mining reward

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