Convergence analysis for gradient descent optimization methods in the training of ReLU neural networks

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Adrian Riekert (University of Münster, Germany)

Wednesday, September 27nd, 2021

Theorem (Hutzenthaler, J, Kruse, Nguyen 2019 *SN PDE*; Kuckuck, J, Padgett 2021)

Let $T, p, \kappa > 0$, *let* $f\colon \mathbb{R} \to \mathbb{R}$ *be Lipschitz,* $\forall\, d \in \mathbb{N}$ *let* $g_d \in C^1(\mathbb{R}^d, \mathbb{R})$ *and* $u_d\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}$ *be an at most poly. grow. solution of*

$$\frac{\partial u_d}{\partial t} = \Delta_x u_d + f(u_d) \qquad \text{with} \qquad u_d(0, \cdot) = g_d,$$

assume $|g_d(x)| + \|(\nabla g_d)(x)\| \leq \kappa d^\kappa (1 + \|x\|^\kappa)$, *let* $\mathcal{A}_l\colon \mathbb{R}^l \to \mathbb{R}^l$, $l \in \mathbb{N}$, *satisfy* $\mathcal{A}_l(x_1, \ldots, x_l) = (\max\{x_1, 0\}, \ldots, \max\{x_l, 0\})$, *let*

$$\mathbf{N} = \cup_{L\in\mathbb{N}} \cup_{l_0,\ldots,l_L\in\mathbb{N}} \left(\times_{n=1}^{L} (\mathbb{R}^{l_n \times l_{n-1}} \times \mathbb{R}^{l_n})\right),$$

let $\mathcal{R}\colon \mathbf{N} \to \cup_{a,b=1}^{\infty} C(\mathbb{R}^a, \mathbb{R}^b)$ *satisfy for all* $L \in \mathbb{N}$, $l_0, \ldots, l_L \in \mathbb{N}$, $\Phi = ((W_1, B_1), \ldots, (W_L, B_L)) \in \times_{n=1}^{L} (\mathbb{R}^{l_n \times l_{n-1}} \times \mathbb{R}^{l_n})$, $x_0 \in \mathbb{R}^{l_0}, \ldots, x_L \in \mathbb{R}^{l_L}$ *with* $\forall\, n \in \{1, \ldots, L\}\colon x_n = \mathcal{A}_{l_n}(W_n x_{n-1} + B_n)$ *that*

$$(\mathcal{R}\Phi)(x_0) = W_L x_{L-1} + B_L,$$

let $\mathcal{P}\colon \mathbf{N} \to \mathbb{N}$ *be the number of parameters, and let* $(G_{d,\varepsilon})_{d\in\mathbb{N},\, \varepsilon\in(0,1]} \subseteq \mathbf{N}$ *satisfy* $\mathcal{P}(G_{d,\varepsilon}) \leq \kappa d^\kappa \varepsilon^{-\kappa}$ *and* $|g_d(x) - (\mathcal{R}G_{d,\varepsilon})(x)| \leq \varepsilon \kappa d^\kappa (1 + \|x\|^\kappa)$. *Then* $\exists\, (U_{d,\varepsilon})_{d\in\mathbb{N},\varepsilon\in(0,1]} \subseteq \mathbf{N}$, $c > 0\colon \forall\, d \in \mathbb{N}, \varepsilon \in (0, 1]\colon$

$$\left[\int_{[0,T]\times[0,1]^d} |u_d(y) - (\mathcal{R}U_{d,\varepsilon})(y)|^p \, dy\right]^{1/p} \leq \varepsilon \quad \text{and} \quad \mathcal{P}(U_{d,\varepsilon}) \leq c\, d^c \varepsilon^{-c}.$$

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Let $T, p, \kappa > 0$, let $f\colon \mathbb{R} \to \mathbb{R}$ be Lipschitz, $\forall\, d \in \mathbb{N}$ let $g_d \in C^1(\mathbb{R}^d, \mathbb{R})$ and $u_d\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ be an at most poly. grow. solution of

$$\frac{\partial u_d}{\partial t} = \Delta_x u_d + f(u_d) \qquad \text{with} \qquad u_d(0,\cdot) = g_d,$$

assume $|g_d(x)| + \|(\nabla g_d)(x)\| \leq \kappa d^\kappa (1 + \|x\|^\kappa)$, let $\mathcal{A}_l\colon \mathbb{R}^l \to \mathbb{R}^l$, $l \in \mathbb{N}$, satisfy $\mathcal{A}_l(x_1, \ldots, x_l) = (\max\{x_1, 0\}, \ldots, \max\{x_l, 0\})$, let

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let $\mathcal{R}\colon \mathbf{N} \to \cup_{a,b=1}^{\infty} C(\mathbb{R}^a, \mathbb{R}^b)$ satisfy for all $L \in \mathbb{N}$, $l_0, \ldots, l_L \in \mathbb{N}$, $\Phi = ((W_1, B_1), \ldots, (W_L, B_L)) \in \times_{n=1}^{L} (\mathbb{R}^{l_n \times l_{n-1}} \times \mathbb{R}^{l_n})$, $x_0 \in \mathbb{R}^{l_0}$, …, $x_L \in \mathbb{R}^{l_L}$ with $\forall\, n \in \{1, \ldots, L\}\colon x_n = \mathcal{A}_{l_n}(W_n x_{n-1} + B_n)$ that

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let $\mathcal{P}\colon \mathbf{N} \to \mathbb{N}$ be the number of parameters, and let $(G_{d,\varepsilon})_{d \in \mathbb{N},\, \varepsilon \in (0,1]} \subseteq \mathbf{N}$ satisfy $\mathcal{P}(G_{d,\varepsilon}) \leq \kappa d^\kappa \varepsilon^{-\kappa}$ and $|g_d(x) - (\mathcal{R}G_{d,\varepsilon})(x)| \leq \varepsilon \kappa d^\kappa (1 + \|x\|^\kappa)$. Then $\exists\, (U_{d,\varepsilon})_{d \in \mathbb{N}, \varepsilon \in (0,1]} \subseteq \mathbf{N}$, $c > 0\colon \forall\, d \in \mathbb{N}, \varepsilon \in (0,1]\colon$

$$\left[\int_{[0,T] \times [0,1]^d} |u_d(y) - (\mathcal{R}U_{d,\varepsilon})(y)|^p \, dy \right]^{1/p} \leq \varepsilon \quad \text{and} \quad \mathcal{P}(U_{d,\varepsilon}) \leq c\, d^c \varepsilon^{-c}.$$

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Theorem (J, Riekert 2021)

Assume $\mu \ll \lambda_{[\mathscr{a},\mathscr{b}]^d}$ *and* $\liminf_{t\to\infty}\|\Theta_t\| < \infty$. *Then there exists* $\vartheta \in (\mathcal{G}_H)^{-1}(\{0\})$ *such that* $\lim_{t\to\infty}\mathcal{L}_H(\Theta_t) = \mathcal{L}_H(\vartheta)$.

Theorem (J, Riekert 2021)

Assume $\mu \sim \lambda_{[\mathscr{a},\mathscr{b}]}$ *with a Lipschitz continuous density, assume that* u *is piecewise affine linear, let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space, for every* $H, k \in \mathbb{N}$, $\gamma \in \mathbb{R}$ *let* $\Theta_n^{H,k,\gamma} \colon \Omega \to \mathbb{R}^{3H+1}$, $n \in \mathbb{N}_0$, *and* $\mathbf{k}_n^{H,k,\gamma} \colon \Omega \to \mathbb{N}$, $n \in \mathbb{N}_0$, *be random variables satisfying for all* $n \in \mathbb{N}_0$ *that*

$$\Theta_{n+1}^{H,k,\gamma} = \Theta_n^{H,k,\gamma} - \gamma\mathcal{G}_H(\Theta_n^{H,k,\gamma}) \quad \text{and} \quad \mathbf{k}_n^{H,k,\gamma} \in \operatorname{arg\,min}_{\ell\in\{1,\dots,k\}} \mathcal{L}_H(\Theta_n^{H,\ell,\gamma}),$$

and assume for all $H \in \mathbb{N}$, $\gamma \in \mathbb{R}$ *that* $\Theta_0^{H,k,\gamma}$, $k \in \mathbb{N}$, *are i.i.d. standard normal. Then* $\exists\, \mathfrak{g} > 0 \colon \forall\, \gamma \in (0, \mathfrak{g}] \colon$

$$\liminf_{H\to\infty}\liminf_{K\to\infty}\mathbb{P}\Big(\limsup_{n\to\infty}\mathcal{L}_H\big(\Theta_n^{H,\mathbf{k}_n^{H,K,\gamma},\gamma}\big) = 0\Big) = 1.$$

Proof based on Fehrman, Gess, J 2020 *JMLR*.

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Proof based on Fehrman, Gess, J 2020 *JMLR*.

Theorem (J, Riekert 2021)

Assume $\mu \ll \lambda_{[a,b]^d}$ *and* $\liminf_{t\to\infty} \|\Theta_t\| < \infty$. *Then there exists* $\vartheta \in (\mathcal{G}_H)^{-1}(\{0\})$ *such that* $\lim_{t\to\infty} \mathcal{L}_H(\Theta_t) = \mathcal{L}_H(\vartheta)$.

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Appendix

Let $d, H, \mathcal{P} \in \mathbb{N}$, $a \in \mathbb{R}$, $b \in (a, \infty)$, $f \in C([a, b]^d, \mathbb{R})$ satisfy $\mathcal{P} = dH + 2H + 1$, let $\mathfrak{R}_r \in C(\mathbb{R}, \mathbb{R})$, $r \in \mathbb{N} \cup \{\infty\}$, satisfy for all $x \in \mathbb{R}$ that $(\bigcup_{r \in \mathbb{N}} \{\mathfrak{R}_r\}) \subseteq C^1(\mathbb{R}, \mathbb{R})$, $\mathfrak{R}_\infty(x) = \max\{x, 0\}$, $\sup_{r \in \mathbb{N}} \sup_{y \in [-|x|, |x|]} |(\mathfrak{R}_r)'(y)| < \infty$, and

$$\limsup_{r \to \infty} \left(|\mathfrak{R}_r(x) - \mathfrak{R}_\infty(x)| + |(\mathfrak{R}_r)'(x) - \mathbb{1}_{(0,\infty)}(x)| \right) = 0,$$

let $\mu \colon \mathcal{B}([a, b]^d) \to [0, \infty]$ be a measure, let $\mathcal{L}_r \colon \mathbb{R}^{\mathcal{P}} \to \mathbb{R}$, $r \in \mathbb{N} \cup \{\infty\}$, satisfy for all $r \in \mathbb{N} \cup \{\infty\}$, $\theta = (\theta_1, \ldots, \theta_{\mathcal{P}}) \in \mathbb{R}^{\mathcal{P}}$ that

$$\mathcal{L}_r(\theta) = \int_{[a,b]^d} \Big(f(x_1, \ldots, x_d) - \theta_{\mathcal{P}} - \sum_{i=1}^{H} \theta_{H(d+1)+i} \Big[\mathfrak{R}_r\big(\theta_{Hd+i} + \sum_{j=1}^{d} \theta_{(i-1)d+j} x_j\big) \Big] \Big)^2 \, \mu(d(x_1, \ldots, x_d)),$$

let $\mathcal{G} \colon \mathbb{R}^{\mathcal{P}} \to \mathbb{R}^{\mathcal{P}}$ satisfy for all $\theta \in \{\vartheta \in \mathbb{R}^{\mathcal{P}} \colon ((\nabla \mathcal{L}_r)(\vartheta))_{r \in \mathbb{N}} \text{ is convergent}\}$ that $\mathcal{G}(\theta) = \lim_{r \to \infty} (\nabla \mathcal{L}_r)(\theta)$, and let $\Theta \in C([0, \infty), \mathbb{R}^{\mathcal{P}})$ satisfy

$$\forall\, t \in [0, \infty) \colon \Theta_t = \Theta_0 - \int_0^t \mathcal{G}(\Theta_s) \, ds.$$

Lemma

There exists an open $U \subseteq \mathbb{R}^{\mathcal{P}}$ such that $\int_{\mathbb{R}^{\mathcal{P}} \setminus U} 1 \, dx = 0$, $(\mathcal{L}_\infty)|_U \in C^1(U, \mathbb{R})$, and $\nabla((\mathcal{L}_\infty)|_U) = \mathcal{G}|_U$.

Let $d, H, \mathcal{P} \in \mathbb{N}$, $\mathscr{a} \in \mathbb{R}$, $\mathscr{b} \in (\mathscr{a}, \infty)$, $f \in C([\mathscr{a}, \mathscr{b}]^d, \mathbb{R})$ satisfy $\mathcal{P} = dH + 2H + 1$, let $\mathfrak{R}_r \in C(\mathbb{R}, \mathbb{R})$, $r \in \mathbb{N} \cup \{\infty\}$, satisfy for all $x \in \mathbb{R}$ that $(\bigcup_{r \in \mathbb{N}} \{\mathfrak{R}_r\}) \subseteq C^1(\mathbb{R}, \mathbb{R})$, $\mathfrak{R}_\infty(x) = \max\{x, 0\}$, $\sup_{r \in \mathbb{N}} \sup_{y \in [-|x|, |x|]} |(\mathfrak{R}_r)'(y)| < \infty$, and

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Let $d, H, \mathcal{P} \in \mathbb{N}$, $\mathscr{a} \in \mathbb{R}$, $\mathscr{b} \in (\mathscr{a}, \infty)$, $f \in C([\mathscr{a}, \mathscr{b}]^d, \mathbb{R})$ satisfy $\mathcal{P} = dH + 2H + 1$, let $\mathfrak{R}_r \in C(\mathbb{R}, \mathbb{R})$, $r \in \mathbb{N} \cup \{\infty\}$, satisfy for all $x \in \mathbb{R}$ that $(\bigcup_{r \in \mathbb{N}} \{\mathfrak{R}_r\}) \subseteq C^1(\mathbb{R}, \mathbb{R})$, $\mathfrak{R}_\infty(x) = \max\{x, 0\}$, $\sup_{r \in \mathbb{N}} \sup_{y \in [-|x|, |x|]} |(\mathfrak{R}_r)'(y)| < \infty$, and

$$\limsup_{r \to \infty} \left(|\mathfrak{R}_r(x) - \mathfrak{R}_\infty(x)| + |(\mathfrak{R}_r)'(x) - \mathbb{1}_{(0,\infty)}(x)| \right) = 0,$$

let $\mu \colon \mathcal{B}([\mathscr{a}, \mathscr{b}]^d) \to [0, \infty]$ be a measure, let $\mathcal{L}_r \colon \mathbb{R}^{\mathcal{P}} \to \mathbb{R}$, $r \in \mathbb{N} \cup \{\infty\}$, satisfy for all $r \in \mathbb{N} \cup \{\infty\}$, $\theta = (\theta_1, \ldots, \theta_{\mathcal{P}}) \in \mathbb{R}^{\mathcal{P}}$ that

$$\begin{aligned} \mathcal{L}_r(\theta) = \textstyle\int_{[\mathscr{a}, \mathscr{b}]^d} \big(f(x_1, \ldots, x_d) \\ - \theta_{\mathcal{P}} - \textstyle\sum_{i=1}^{H} \theta_{H(d+1)+i} \big[\mathfrak{R}_r(\theta_{Hd+i} + \textstyle\sum_{j=1}^{d} \theta_{(i-1)d+j} x_j) \big] \big)^2 \, \mu(\mathrm{d}(x_1, \ldots, x_d)), \end{aligned}$$

let $\mathcal{G} \colon \mathbb{R}^{\mathcal{P}} \to \mathbb{R}^{\mathcal{P}}$ satisfy for all $\theta \in \{\vartheta \in \mathbb{R}^{\mathcal{P}} \colon ((\nabla \mathcal{L}_r)(\vartheta))_{r \in \mathbb{N}} \text{ is convergent}\}$ that $\mathcal{G}(\theta) = \lim_{r \to \infty} (\nabla \mathcal{L}_r)(\theta)$, and let $\Theta \in C([0, \infty), \mathbb{R}^{\mathcal{P}})$ satisfy

$$\forall\, t \in [0, \infty) \colon \Theta_t = \Theta_0 - \textstyle\int_0^t \mathcal{G}(\Theta_s) \, \mathrm{d}s.$$

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1. *there exist infinitely many non-global local minima of* $\mathcal{L}_\infty$,
2. *there exist infinitely many saddle points of* $\mathcal{L}_\infty$, *and*
3. *it holds that* $\limsup_{t\to\infty} \mathcal{L}_\infty(\Theta_t) = 0$.

Theorem (J Riekert 2021)

Assume $\sup_{t\in[0,\infty)}\|\Theta_t\| < \infty$ *and* $\mu \ll \lambda_{[\mathscr{a},\mathscr{b}]^d}$. *Then there exists* $\vartheta \in \mathcal{G}^{-1}(\{0\})$ *such that* $\limsup_{t\to\infty} \mathcal{L}_\infty(\Theta_t) = \mathcal{L}_\infty(\vartheta)$.

Theorem (Cheridito J Riekert Rossmannek 2021, J Riekert 2021)

Assume for all $x, y \in [\mathscr{a}, \mathscr{b}]^d$ *that* $f(x) = f(y)$. *Then*

1. *there exist no non-global local minima of* $\mathcal{L}_\infty$,
2. *there exist no saddle points of* $\mathcal{L}_\infty$,
3. *it holds for all* $\theta \in \mathcal{G}^{-1}(\{0\})$ *that* θ *is a global minima of* $\mathcal{L}_\infty$, *and*
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d	Mean	Std. dev.	Ref. value	rel. L^1-error	Std. dev. rel. error	avg. runtime
10	2.07017	0.00634850	2.04629	0.01167	0.00310245	58.200
50	3.15098	0.00275839	3.13788	0.00417	0.00087906	58.359
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Approximations for $u(1,0)$; Time steps: 24;

SGD steps: 500 ($d < 10^4$), 600 ($d = 10^4$); Learning rates: $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$

Beck, Becker, Cheridito, J, Neufeld 2019 *SISC* (to appear)

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10	2.07017	0.00634850	2.04629	0.01167	0.00310245	58.200
50	3.15098	0.00275839	3.13788	0.00417	0.00087906	58.359
100	3.75329	0.00136920	3.74471	0.00229	0.00036564	58.329
200	4.46734	0.00079688	4.46172	0.00126	0.00017860	58.159
300	4.94586	0.00087736	4.94105	0.00097	0.00017756	58.819
500	5.62126	0.00045092	5.61735	0.00070	0.00008027	57.670
1000	6.68594	0.00040334	6.68335	0.00039	0.00006035	66.546
5000	9.97266	0.00047098	9.99835	0.00257	0.00004711	393.894
10000	11.87860	0.00022705	11.89099	0.00104	0.00001909	1687.680

Approximations for $u(1, 0)$; Time steps: 24;

SGD steps: 500 ($d < 10^4$), 600 ($d = 10^4$); Learning rates: $\frac{1}{10}, \frac{1}{100}, \frac{1}{1000}$

Beck, Becker, Cheridito, J, Neufeld 2019 *SISC* (to appear)

Hamiltonian-Jacobi-Bellman equations

Consider

$$\frac{\partial u}{\partial t} = \Delta_x u - \|\nabla_x u\|^2$$

with $u(0, x) = \sqrt{\|x\|}$ for $t \in [0, 1]$, $x \in \mathbb{R}^d$.

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Full history recursive Multilevel-Picard method: Let $T > 0, L, p \geq 0, \Theta = \cup_{n=1}^{\infty} \mathbb{Z}^n$, $\forall\, d \in \mathbb{N}$ let $g_d \in C(\mathbb{R}^d, \mathbb{R})$ satisfy $\forall\, x \in \mathbb{R}^d \colon |g_d(x)| \leq L(1 + \|x\|^p)$, let $f \colon \mathbb{R} \to \mathbb{R}$ be Lipschitz, let $(\Omega, \mathcal{F}, \mathbb{P})$ probab. sp., let $W^{d,\theta} \colon [0, T] \times \Omega \to \mathbb{R}^d$, $d \in \mathbb{N}$, $\theta \in \Theta$, be i.i.d. Brownian motions, let $S^\theta \colon [0, T] \times \Omega \to \mathbb{R}$, $\theta \in \Theta$, i.i.d. continuous satisfying $\forall\, t \in [0, T]$, $\theta \in \Theta$ that S^θ_t is $\mathcal{U}_{[t,T]}$-distributed, assume that $(S^\theta)_{\theta \in \Theta}$ and $(W^{d,\theta})_{\theta \in \Theta, d \in \mathbb{N}}$ are independent, let $U^{d,\theta}_{n,M} \colon [0, T] \times \mathbb{R}^d \times \Omega \to \mathbb{R}$, $d, n, M \in \mathbb{Z}$, $\theta \in \Theta$, satisfy $\forall\, d, M \in \mathbb{N}$, $n \in \mathbb{N}_0$, $\theta \in \Theta$, $t \in [0, T]$, $x \in \mathbb{R}^d$:

$$
\begin{aligned}
U^{d,\theta}_{n,M}(t, x) &= \sum_{m=1}^{M^n} \frac{g_d(x + W^{d,(\theta,0,-m)}_{T-t}) \mathbb{1}_{\mathbb{N}}(n)}{M^n} \\
&\quad + \Bigg[\sum_{l=0}^{n-1} \tfrac{(T-t)}{M^{n-l}} \sum_{m=1}^{M^{n-l}} f\Big(U^{d,(\theta,l,m)}_{l,M}\big(S^{(\theta,l,m)}_t, x + W^{d,(\theta,l,m)}_{S^{(\theta,l,m)}_t - t}\big)\Big) \\
&\qquad - \mathbb{1}_{\mathbb{N}}(l)\, f\Big(U^{d,(\theta,l,m)}_{l-1,M}\big(S^{(\theta,l,m)}_t, x + W^{d,(\theta,l,m)}_{S^{(\theta,l,m)}_t - t}\big)\Big) \Bigg]
\end{aligned}
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and $\forall\, d, n \in \mathbb{N}$ let $\mathrm{Cost}_{d,n} \in \mathbb{N}$ be the comp. cost of $U^{d,0}_{n,n}(0, 0)$.

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+ \Bigg[\sum_{l=0}^{n-1} \frac{(T-t)}{M^{n-l}} \sum_{m=1}^{M^{n-l}} f\Big(U^{d,(\theta,l,m)}_{l,M}\big(S^{(\theta,l,m)}_t, x + W^{d,(\theta,l,m)}_{S^{(\theta,l,m)}_t - t}\big)\Big) \\
- \mathbb{1}_{\mathbb{N}}(l)\, f\Big(U^{d,(\theta,-l,m)}_{l-1,M}\big(S^{(\theta,l,m)}_t, x + W^{d,(\theta,l,m)}_{S^{(\theta,l,m)}_t - t}\big)\Big) \Bigg]
\end{aligned}
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&\quad + \Bigg[\sum_{l=0}^{n-1} \frac{(T-t)}{M^{n-l}} \sum_{m=1}^{M^{n-l}} f\Big(U_{l,M}^{d,(\theta,l,m)}\big(S_t^{(\theta,l,m)}, x + W_{S_t^{(\theta,l,m)}-t}^{d,(\theta,l,m)}\big)\Big) \\
&\qquad - \mathbb{1}_{\mathbb{N}}(l)\, f\Big(U_{l-1,M}^{d,(\theta,-l,m)}\big(S_t^{(\theta,l,m)}, x + W_{S_t^{(\theta,l,m)}-t}^{d,(\theta,l,m)}\big)\Big) \Bigg]
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\end{aligned}
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Then

(i) $\forall\, d \in \mathbb{N}\colon$ there exists $u_d\colon [0,T] \times \mathbb{R}^d \to \mathbb{R}$ at most polyn. grow. solution of

$$\tfrac{\partial u_d}{\partial t} + \tfrac{1}{2}\Delta_x u_d + f(u_d) = 0 \qquad \text{with} \qquad u_d(T,\cdot) = g_d$$

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(ii) $\forall\, \delta > 0\colon$ there exist $n\colon \mathbb{N} \times (0,\infty) \to \mathbb{N}$ and $C > 0\colon$ $\forall\, d \in \mathbb{N}, \varepsilon > 0\colon$

$$\left(\mathbb{E}\Big[\big|u_d(0,0) - U^{d,0}_{n_{d,\varepsilon},n_{d,\varepsilon}}(0,0)\big|^2\Big]\right)^{1/2} \leq \varepsilon$$

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Extensions: Algorithms/Simulations/Proofs: Fully nonlinear PDEs (Beck, E, J 2018 *JNS*), Optimal stopping (Becker, Cheridito, J 2018 *JMLR*), Uniform errors (Beck, Becker, Grohs, Jaafari, J 2018), Semilinear PDEs/CVA (Hutzenthaler, J, von Wurstemberger 2019 *EJP*, Hutzenthaler, J, Kruse, Nguyen 2020), Nonlipschitz nonlinearities (Beck, Hornung, Hutzenthaler, Jentzen, Kruse 2019 *J. Numer. Math.*), Gradient dependent nonlinearities (Hutzenthaler, J, Kruse 2019), Elliptic PDEs (Beck, Gonon, J 2020), Discrete problems (Beck, J, Kruse 2020), . . .

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