Dynamic Risk Management for sustainable Savings and Retirement

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1 Introduction

As the population ages with rapidly changing environment over the past decade, a new reality of long term Savings and Retirement has been emerging, as follows:

- Increased life expectancy
  - Retirees live longer (50% > 92 yrs, 25% > 97 yrs), which needs for a 30+ year plan
  - Need for insurance against longevity risk
- Changing retirement income sources
  - Reduction in Social Security to meet retirement income needs
  - Increased reliance on Individual Savings Plans
- Need for return and secure products
  - While 2% inflation over 25 years lead to 40% reduction in purchasing power
  - And 1% greater return means 10 extra years of income
  - Persistent zero interests bonds *vs.* equity significant dividends or market growth
  - While increasingly recurrent market crashes

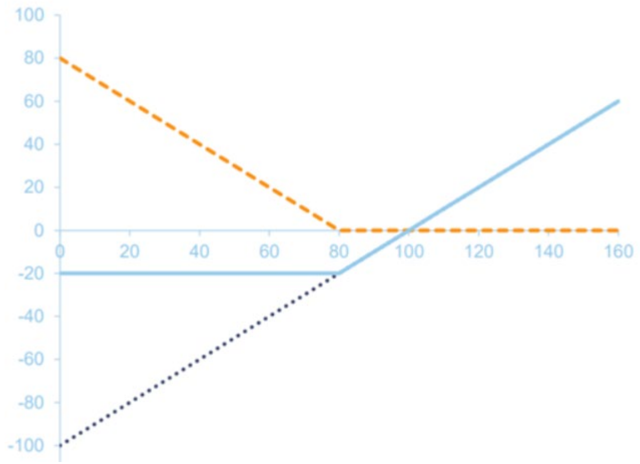
As a result sustainable long term savings and retirement need markets upside potential, combined with downside protection through partial or full capital guarantees post a waiting period. Two major types of long term savings and retirement investments are available:

- The traditional one is the "Euro fund" where the policyholder pays a premium for stable low Bonds returns combined with capital guarantee. Unfortunately the persistent close to zero bonds yields makes this opportunity neither profitable for the policyholder nor sustainable for the insurer due its very high cost of capital within a zero interest rates environment.

- The alternative lies in "Long Term Equity Guaranteed Investments", where the policyholder looks for significant upside exposure (through the Equity investment) and pays a premium to ensure downside protection.

Given the downside protection is subject to financial market downturns, the insurance company may suffer a loss in case of financial market turmoil, as illustrated by the following example of a product characterized by a partial guaranteed lump sum amount post a waiting period: a policyholder invests €100 in a given Equity fund and the insurer guarantees him a minimum of €80 at a 1-year horizon.

Figure 1

<table>
<tr><th colspan="2">Equity Savings & Retirement Guaranteed Product decomposed into an Equity Fund + Protective Equity Put</th></tr>
<tr><td>
</td><td>The payoff for the client is given by the blue line (which can be decomposed b/w the fund – dotted blue – and a Put option – dotted red)

• If the Equity fund terminal value is above €100, then the client makes a profit

• If the Equity fund terminal value is b/w €80 and €100, then the client suffers some loss (b/w 0 and €20)

• If the Equity fund terminal value is below the guarantee level €80, then the loss for the client is capped at €20, while the insurer pays the difference between the terminal value and €80, which can be significant, which is called “Put option payoff”. This risk of loss to the insurer must be taken into account in the insurance premium at contract’s inception, as future potential loss cannot be passed on to the customer at a later date.

From then on, the customer's gain, which must be included in the insurance premium, is broken down into: action + "Put".</td></tr>
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In order to mitigate those financial risks borne by the insurer, such a partial downside protection has to be “hedged” by the insurer, by investing in “hedge assets” (*e.g.* Futures, options) whose variations with respect to economic variables offset the variation of the downside protection embedded within the savings & retirement product – and whose expected cost has to be integrated into the insurance premium. As a matter of fact, insurance companies are major users of “hedge” assets to protect the value of their large Asset under Management (the Individual Savings business amounts to ~€1,800 bn in France, ~$6,000 bn in the US), as illustrated by the significant increase their notionals from $786 billion as of 2010 to $2,300 billion as of FY 2018, out of which $1.1 trillion are invested in Equity hedges).

As the Long Term Equity Guaranteed Investment can be decomposed into the Equity Asset and a Put option, hedging the risks embedded within the downside protection can be done through buying Put options available in limited quantity in the financial market. However, buying a large number of Equity

Put options may bear pressures on the unit price as Equity Put options are highly sensitive to supply and demand balance, which implies a higher cost of hedging than initially expected.

There indeed have been significant evidence of growing costs stemming from supply-and-demand imbalance for options, driven by the hedging activity of large players, in particular during equity selloffs with growing likelihood over the past decade (May 2010, August 2011, August 2015, January 2016, June 2016, February 2018, October-December 2018, March-April 2020).

Besides, recent regulatory frameworks (*e.g*., Solvency II in Europe, NAIC reform and US GAAP LDTI in the US) force insurers to hold sufficient capital requirements offsetting the economic (mark-to-market) risks embedded within those downside protection guarantees, in order to remain solvent during periods of market stress. Those capital requirements can be reduced if the economic risks are mitigated through buying put options, whose prices will increase as a result of supply-and-demand imbalance.

As a result, the cost of placing one large order to close a position becomes significantly greater than the sum of infinitely small orders differed in time. For this reason, an explicit modeling of the market impact function driven by the transaction size is required (which is not considered by traditional pricing models). Such market impact function depends on the "temporary impact strength" that is proportional to the main empirically observed drivers: the speed of option trading (*i.e.,* number of options per unit of time), the equity stock level and the option sensitivity to the equity stock.

In this context, the best order execution (*i.e.* minimizing the market impact) cannot be defined as a single trade, but turns out to be the sequence of small trades over the course of several days that optimizes a target. Minimizing market impact will be beneficial not only to the insurer through a lower hedging cost but also to the customer through a lower price.

Such an order execution strategy will also depend on the insurer's risk appetite (*e.g.* minimization of the mean cost or of the mean variance that also penalizes the dispersion of the profit and loss). The standard procedure of the Hamilton-Jacobi-Bellman (HJB) framework in stochastic control problems is applied, coupled with numerical schemes.

In case the Equity Put option becomes too expensive (due to higher market uncertainties or too large market impact), the insurer will replicate them synthetically and dynamically through Equities (or Equity Futures) that benefit from lower transaction costs. Such a synthetic replication needs to be dynamic in order for linear Equities to piecewise match convex shapes of the Equity Puts as the market moves, which provides higher replication costs and optimal transaction size resulting from the "feedback" market impact of the large Equities transaction size on the Equity asset price.

In section 2 the Equity Put price consistent with market impact is modelled and best order execution strategies consistent with the insurer's risk appetite are illustrated. In section 3 the proxy synthetic replication through using a large quantity of Equity assets is modelled within a specific No Arbitrage Framework, providing a higher cost of replication and larger execution order size as compared with no market impact.

2 Dynamic risk management reduces the cost to the insurer and the price to the insured, by minimizing market impact with respect to the insurer's risk aversion

2.1. Equity Puts pricing model including market impact

The model below is inspired from Leland's option replication with transaction costs (see [9]), where the market impact is incorporated into the volatility of the asset σ as a market impact function f (dependent on time, volatility, inventory and trading rate).

$$\tilde{\sigma}^2 = \sigma^2 + f(t, \dot{x}_t, x_t, \sigma)$$

We follow the approach by Almgren (see [10] and [4])) where the market impact function is a combination of two components: a permanent component that reflects the information transmitted to the market by the buy/sell imbalance, and a temporary component that reflects the price concession needed to attract counterparties within a specified short time interval. We adapt such approach to derivatives through the "**enlarged volatility"** expression as follows:

$$\tilde{\sigma}_t^2 = \sigma^2 + (\tilde{\eta}\dot{x}_t + \tilde{\gamma}(x_t - x_0))\sqrt{\hat{T} - t}\sigma$$ where $\tilde{\eta} = \eta\sqrt{\frac{8}{h\pi}}$ and $\tilde{\gamma} = \gamma\sqrt{\frac{8}{h\pi}}$.

where η and γ are constants. The number of shares is x(t), while $\dot{x}_t$ (its derivative with regards to time) is the the speed of trading of the security. The term $\eta\dot{x}_t$ corresponds to the temporary or instantaneous impact of trading $\dot{x}_t dt$ shares at time t which only affects this current order. The term $\gamma(x_t - x_0)$ is the permanent price impact that was accumulated by all transactions until time t.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the usual probability space on the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$. In the absence of market impact and under a zero risk-free rate, the no-arbitrage price of a put option is defined by $P_t = \mathbb{E}_{\mathbb{Q}}[(K - S_{\hat{T}})^+|\mathcal{F}_t]$ under the risk-neutral probability measure Q under which the discounted asset price is a martingale.

The Equity Put option price including market impact is then expressed through a Black-Scholes-like partial differential equation using the "enlarged volatility" illustrative of the market impact, where buying the option will typically lead to increasing its price. The higher the trading speed and quantity, the higher the volatility thus the option price:

$$\begin{cases} \partial_u \tilde{P}(u,S) + \frac{1}{2}\tilde{\sigma}_t^2 S^2 \partial_{SS}\tilde{P}(u,S) = 0, \quad (u,S) \in [t, \hat{T}[\times]0, \infty] \\ \tilde{P}(\hat{T}, s) = (K - s)^+. \end{cases}$$

Using a simple Taylor approximation to the first order, the expression can be expressed as a sum of the Black-Scholes option price without market impact and an additional term corresponding to the option market impact:

$$\tilde{P}(t,S_t) \approx P(t,S_t) + (\tilde{\sigma}_t^2 - \sigma^2)\partial_v P(t,S_t) \approx P(t,S_t) + \frac{1}{2}\{\tilde{\eta}\cdot\dot{x}_t + \tilde{\gamma}\cdot(x_t - x_0)\}\cdot\sqrt{\hat{T} - t}v(t,S_t)$$

where $\nu(t,S_t) = \partial_\sigma P$ is the Black-Scholes "vega" of the option:

$$\nu(t,S_t) = \sqrt{\hat{T}-t}S_tN'(d_1) = \sqrt{\hat{T}-t}KN'(d_2)$$

$$N'(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2},$$

$$d_1 = \frac{\log\frac{S_t}{K}+\frac{1}{2}\sigma^2(\hat{T}-t)}{\sigma\sqrt{\hat{T}-t}},$$

$$d_2 = \frac{\log\frac{S_t}{K}-\frac{1}{2}\sigma^2(\hat{T}-t)}{\sigma\sqrt{\hat{T}-t}} = d_1 - \sigma\sqrt{\hat{T}-t}.$$

For simplicity of notation the permanent impact is excluded in the following sections (i.e., $\tilde{\gamma}=0$). Using the "Vega-Gamma" relationship (Gamma is the second order sensitivity of the option price with respect to the underlying $\frac{d^2P}{d^2S}$, which is equal to the optimal change in the quantity of assets required to hedge the Put).

$\nu = \sigma\tau S^2\Gamma$, where $\Gamma(t,S_t) = \frac{N'(d_1)}{S_t\sigma\sqrt{\hat{T}-t}} = \frac{KN'(d_2)}{S_t^2\sigma\sqrt{\hat{T}-t}}.$

the price including market impact is then:

$$\tilde{P}_t = P_t + \frac{1}{2}\tilde{\eta}\dot{x}_t\sigma S_t^2(\hat{T}-t)^{3/2}\Gamma(t,S_t).$$

2.2. The Optimal Order Execution minimizes market impact with respect to the insurer's Risk Appetite

The optimal dynamic order execution strategy unfolds over the course of several days [0,T], consistent with changing market conditions and according market impact. Let us consider a buying order execution strategy x(t) in which an amount X of options with fixed strike K and maturity T are bought by a fixed time horizon [0,T] with the conditions x(0) = X and x(T) = 0.

At each time t, $\dot{x}_t dt$ options are bought at price $\tilde{P}_t$ which is the option impact price defined by the price equation above. Thus, the cost arising from the strategy x is $\mathcal{C}(x) := \int_0^T \tilde{P}_t\dot{x}_t dt$

$$\mathcal{C}(x) = \int_0^T P_t\dot{x}_t dt + \frac{1}{2}\tilde{\eta}\int_0^T \dot{x}_t^2\sigma S_t^2(\hat{T}-t)^{3/2}\Gamma(t,S_t)dt,$$

$$\mathcal{C}(x) = -XP_0 - \int_0^T \sigma x_t S_t\Delta(t,S_t)dW_t + \frac{1}{2}\tilde{\eta}\sigma\int_0^T \dot{x}_t^2 S_t^2(\hat{T}-t)^{3/2}\Gamma(t,S_t)dt,$$

where $\Delta = \frac{dP}{dS}$ is the 1st order sensitivity of the option price with respect to the underlying, so called Black-Scholes "delta" of the option, which is equal to the quantity of assets required to hedge the Put.

The insurer's objective is then to minimize a certain cost objective function, which may take into account his risk aversion. Here we will consider two risk appetite cases:

- the mean cost $\mathbb{E}[\mathcal{C}(x)]$, which corresponds to the risk-neutral case (the insurer has no risk aversion)
- the mean-variance cost $\mathbb{E}[\mathcal{C}(x)] + \lambda \mathbb{V}ar[\mathcal{C}(x)]$, *i.e.* a risk/reward criterion, (which includes the mean case if λ = 0), where λ is the variance penalty

2.2.1. Optimal Order Execution minimizing market impact with no insurer's risk aversion

The mean cost is usually used for an agent who does not monitor the risk of his strategy:

$$\mathbb{E}[\mathcal{C}(x)] = -XP_0 + \frac{1}{2}\tilde{\eta}\mathbb{E}\Big[\int_0^T \dot{x}_t^2 S_t^2 (\hat{T} - t)^{3/2} \Gamma(t, S_t)\, dt\Big].$$

Theorem The optimal transaction size and pace strategies resulting in minimizing the mean cost under the Black-Scholes framework is illustrated in Figure 2 and characterized by

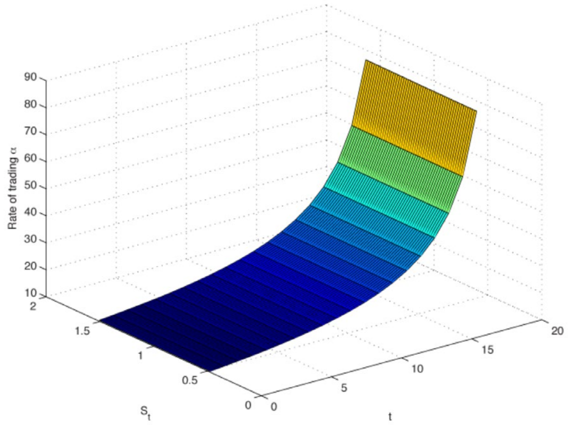
$$\dot{x}^*(t) = \frac{K_1}{(\hat{T} - t)^{3/2}},$$

$$x^*(t) = \frac{K_1}{(\hat{T} - t)^{1/2}} + K_2$$ where $K_1 = \frac{X}{2(\hat{T}^{-\frac{1}{2}} - (\hat{T} - T)^{-\frac{1}{2}})}$ and $K_2 = -2K_1(\hat{T} - T)^{-1/2}$

Proof: see [8]

Figure 2

Optimal Rate of Trading depending on stock level and time passing



Under the mean cost case, the optimal order execution strategy provides a rather stable pace of trading, depending only mildly on the stock price path. Besides, the insurer must acquire at a gradual faster rate as time passes, as expected given the fixed quantity to buy within a fixed time period.

2.2.1. Optimal Order Execution minimizing market impact with risk aversion

We will now develop the optimal order execution framework under the mean-variance case (see Forsyth (2013) [12], Almgren (2011) [10]), where the optimal strategy turns out to be more sensitive to the underlying price evolution. Actually the mean-variance objective function can be approximated as:

$$E\left[C(x)\right]+\lambda Var\left[C(x)\right]\approx E\left[\int_0^T \frac{1}{2}\tilde{\eta}\sigma\dot{x}_t^2 S_t^2(\hat{T}-t)^{\frac{3}{2}}\Gamma(t,S_t)dt+\tilde{\lambda}\int_0^T x_t^2\sigma^2 S_t^2\Delta^2(t,S_t)dt\right]$$

We then set up the dynamic programming problem where we parameterize as before the strategies x by their trading speed or trading rate α defined as $-\dot{x}_t$: $x_t^\alpha := X - \int_0^t \alpha_s ds,\quad 0\le t\le T$.

We restrict our framework to a Markovian trading rate (*i.e.,* the agent's optimal trading speed at time t is completely determined by the current state). Using the standard procedure of deriving the Hamilton-Jacobi-Bellman equation in stochastic control problems, the solution to the optimization problem solves the following PDE:

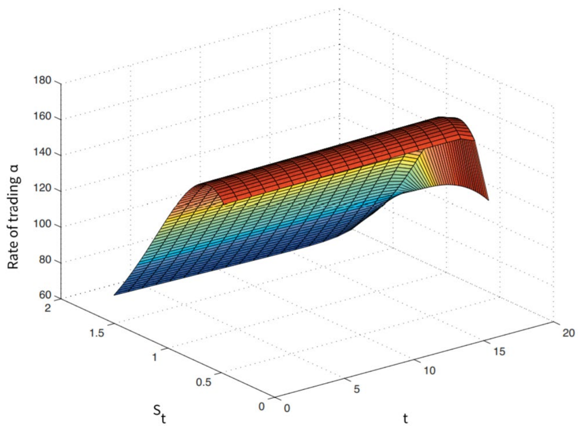
$$\partial_t U+\frac{1}{2}\sigma^2 S^2\partial_{SS}U+\tilde{\lambda}x^2S^2\Delta^2(t,S)+\inf_{\alpha\in\mathbb{R}}\left\{\alpha^2S^2\left(\hat{T}-t\right)^{3/2}\Gamma(t,S)-\alpha\partial_x U\right\}=0$$

combined with the so-called finite-fuel constraint (*i.e.,* $\int_0^T\alpha_t dt = X$) .

Although this minimization problem does not admit a closed-form solution, this quasi-linear PDE can be solved numerically using finite differences methods. Figure 3 illustrates the optimal order execution strategy through the rate of trading as a function of the underlying price S and time t.

Figure 3

Optimal Rate of Trading depending on stock level and time passing



Note: Mean objective (λ = 0, top left) or mean-variance (λ = 1, top right; 10, bottom left; 100, bottom right)

In contrast to the mean cost case, as the risk aversion (dispersion) λ increases, the optimal order execution increasingly depends on the stock path, with a faster pace as the stock level decreases, consistent with a higher put option cost, which implies to trade as soon as possible. Besides, as time passes the trading pace decreases, as the mean-variance prevents the insurance company from waiting too long to buy a large quantity.

3 The potential lack of Puts calls for a dynamic synthetic replication that minimizes the "feedback" effects of the stock sales on the cost of replication

3.1. The pricing model of synthetic replication with market impact

In case the Equity Put options become too expensive due to higher market uncertainties or too large market impact, or just unavailable due to the long term maturities of retirement products, the insurer will replicate them synthetically and dynamically through Equities (or Equity Futures) benefiting from lower transaction costs.

The quality of such a piecewise synthetic approximation of the Equity Puts lies in the ability to dynamically fit the convex shape of the Equity Put by a series of subsequent dynamic trades in linear Equity assets as the Equity market changes, which involves a regularly updated quantity of Equity assets, equal to the sensitivity of the Equity Put to the Equity underlying (so-called “delta”).

Unfortunately, due to market impact associated with the large Equities transaction size, such a dynamic sequence of Equity trades involves some market inefficiencies (so-called “feedback effects”), as the insurer sells Equity as the market drops or buys Equity as the market rallies, which exacerbates the Equity market shocks and increases the replication cost, as illustrated over the past decade (May 2010, August 2011, August 2015, January 2016, June 2016, February 2018, October-December 2018, March-April 2020).

The objective of the optimal order replication strategy is to minimize such market impact. It is determined by a no arbitrage framework that incorporates the impact of the large trader’s large transactions on Equities. We consider here the interaction of one “large trader” whose action affects prices *vs.* many price-takers or “small traders”; the usual “no arbitrage” condition doesn’t apply. We use a continuous time version of Jarrow’s “no market manipulation strategies” (see Jarrow (1994) [5] and Bierbaum (1997) [1]), which requires additional assumptions:

- The asset price is independent of the large trader’s past holdings
- Real wealth (as if the holdings were liquidated)
- Synchronous markets condition
- Prices adjust instantaneously across underlying and derivatives
- Absence of corners

The dynamics of the asset price return with impact can be modeled as a modified version of the usual dynamics (under zero interest rates assumption)

$$\frac{dS_t}{S_t} = \sigma_t dW_t + \rho_t d\alpha_t$$

Where

- $\sigma_t \mathrm{dW_t}$ is the dynamics of the asset return without market impact,
- $\rho_t \mathrm{d}\alpha_t$ is the specific market impact contribution, where α is the transaction size in assets, while ρ represents the intensity of the market impact (*e.g.,* the ratio of change in the price of the underlying to the quantity traded). So $\frac{1}{\rho_t S_t}$ represents the market depth at time t, (the quantity of assets required to move prices by one unit).

Under a zero risk-free interest rate (for simplicity of notation), the "No Arbitrage" cost of synthetic replication of a Put is the expected discounted future payoff under the associated martingale measure $\mathbb{Q}'$, $P_t = \mathbb{E}_{\mathbb{Q}'}[\{(K - S_T)^+ | \mathcal{F}_t\}]$, which provides the following pricing equation as we now apply the Black-Scholes continuous-time delta-hedging replication framework to the modified asset return dynamics under $\mathbb{Q}'$ [2](see Platen (1998) [11], Frey (1998) [3])

$$\begin{cases} \frac{dP}{dt} + \frac{1}{2}\left(\frac{\sigma S}{1 - \varrho S \frac{d^2P}{d^2S}}\right)^2 \frac{d^2P}{d^2S} = 0 \\ P_T = n(K - S_T)^+ \end{cases}$$

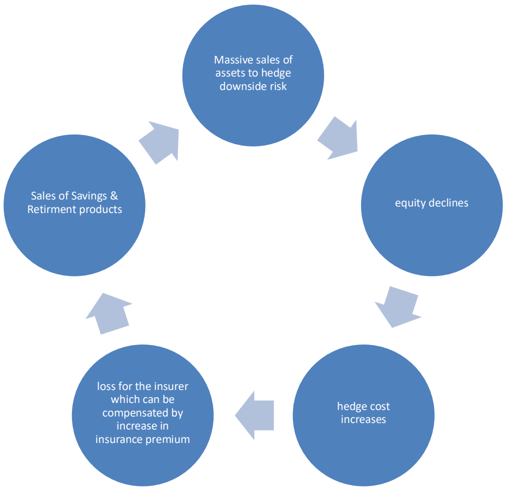
where n is the number of Equity Put options to hedge. Such a cost including market impact is equivalent to the cost without market impact but with so-called "feedback" volatility as below, increasing with both the market impact intensity ρ and the increase in the sales of assets required hedging the Put:

$$\tilde{\sigma} = \frac{\sigma}{\left(1 - \varrho S \frac{d^2P}{d^2S}\right)}$$

This modified Black-Scholes equation is a fully nonlinear parabolic PDE, requiring specific numerical implementation ensuring accuracy, flexibility and stability (see Touzi (2011) [2] and Kalife (2012) [6]).

3.2. Optimal synthetic replication with market impact minimizes the "feedback effects" stemming from the stock sales on the replication cost

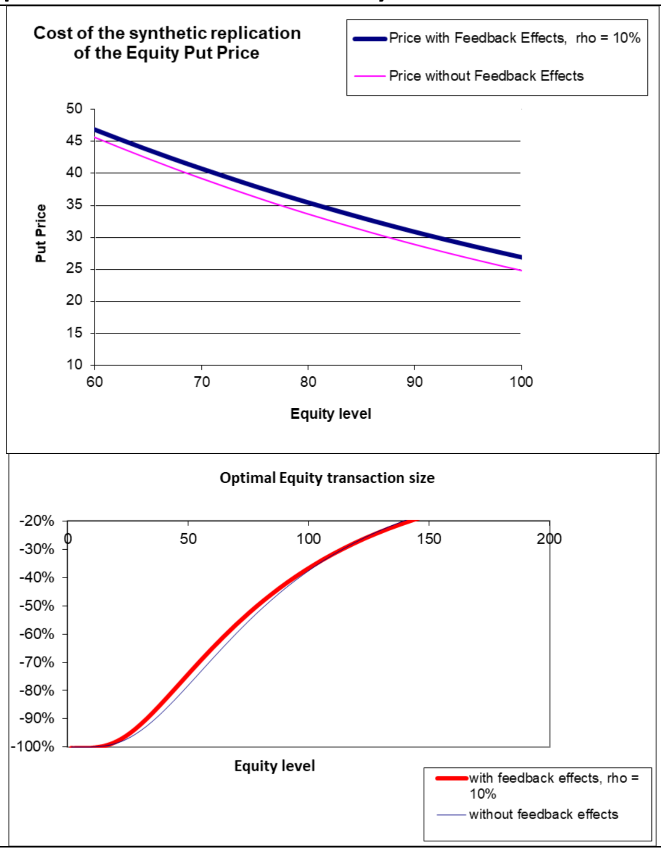
Actually, as the insurer synthetically replicates the long Equity Put position by selling a quantity of stocks equal to the sensitivity of the Put price to the stock, whenever market falls she has to sell further, which as a "feedback effect" makes the stock price fall further, compounded by a larger volatility thus a larger replication cost.



As illustrated in Figure 4, the synthetic dynamic strategy presented above enables to minimize such impact into an optimal replication cost that depends on the market impact intensity ("rho"). The associated optimal dynamic strategy in stocks is provided by the sensitivity of such optimal replication cost to the Equity level as illustrated further below.

Figure 4

Optimal replication cost and associated dynamic transaction size in Equities



4 Conclusion

Within the context of rapidly ageing population with increasing life expectancy, the reduction in Social Security to meet retirement income needs, the zero interest rates environment compounded with recurrent market crashes, sustainable long term savings and retirement need Equity markets upside combined with some capital guarantees post a waiting period, thus embedding Equity Puts-like downside protection.

Given the downside protection is subject to financial market downturns, insurance companies need to hedge financial risks though buying large quantities of Equity Puts, thus incurring market impact translating into higher hedging costs than expected. Dynamic Risk Management provides a best order execution strategy that minimizes the market impact thus the hedging cost with respect to the insurer's risk appetite, which also benefits to the customer through a lower price.

In case the Equity Put option becomes too expensive or illiquid, the insurer replicates them synthetically and dynamically through selling large Equities transactions reinforcing Equity market declines, which provides larger replication costs. Dynamic Risk Management provides a best order execution strategy that minimizes such impact on replication costs through optimal dynamic equity selling transactions depending on the Equity levels.

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