Riding on the Smiles

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1 Extended Abstract

This paper investigates the calibration performance of several multifactor stochastic volatility models. There is an empirical evidence that the dynamics of the implied volatility surface is driven by several factors, see Skiadopoulos et al. (1999). This leads to the extensions of the seminal Heston stochastic volatility model proposed by Da Fonseca et al. (2008b) and Christoffersen et al. (2007). In an arbitrage-free frictionless financial market we consider a risky asset whose price follows:

$$\frac{dS_t}{S_t} = (r-\delta)dt + \mathrm{Tr}\left[\sqrt{\Sigma_t}\left(dW_t R^\top + dB_t\sqrt{\mathbb{I} - RR^\top}\right)\right], \tag{1}$$

where r denotes the (not necessarily constant) risk-free interest rate, Tr is the trace operator, $W_t, B_t \in M_n$(the set of square matrices) are composed by n^2 independent Brownian motions under the risk-neutral measure (B_t and W_t are independent), $R \in M_n$ represents the correlation matrix (identified up to a rotation) and Σ_t belongs to the set of symmetric $n \times n$ positive-definite matrices (as well as its square root $\sqrt{\Sigma_t}$). From (1), it follows that the quadratic variation of the risky asset is the trace of the matrix Σ_t: that is, in this specification the volatility is multi-dimensional and depends on the elements of the matrix process Σ_t, which is assumed to satisfy the following dynamics:

$$d\Sigma_t = \left(\Omega\Omega^\top + M\Sigma_t + \Sigma_t M^\top\right)dt + \sqrt{\Sigma_t}dW_t Q + Q^\top\left(dW_t\right)^\top\sqrt{\Sigma_t}, \tag{2}$$

with $\Omega, M, Q \in M_n$, Ω invertible.
Equation (2) characterizes the Wishart process introduced by Bru (1991) and then by Gourieroux and Sufana (2004), Grasselli and Tebaldi (2008), Da Fonseca et al. (2008a),Da Fonseca et al. (2008b), Da Fonseca et al. (2007), and represents the matrix analogue of the square root mean-reverting process. In order to grant the strict positivity and the typical mean reverting feature of the volatility, the matrix M is assumed to be negative semi-definite, while Ω satisfies $\Omega\Omega^\top = \beta Q^\top Q$ with the real parameter $\beta > n-1$ (see Bru (1991) p. 747).
We shall see that within the Wishart specification, analytical tractability is preserved exactly as in the (1-dimensional) Heston model. In fact, it is well known that in order to solve the pricing problem of plain vanilla options, it is enough to compute the conditional characteristic function (under the risk-neutral measure) of the underlying forward price (see e.g. Duffie, Pan and Singleton 2000) or equivalently of the return process $Y_t = \ln F_{t,T}$
Using a data set of derivatives on the major indices we study the calibration properties of these models using the FFT as the pricing methodology (Carr and Madan (1999)). We also study if adding jumps improves significantly the calibration accuracy of the models.

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Then we focus on basket option pricing models and more precisely on the WASC model (Wishart Affine Stochastic Correlation) proposed by Da Fonseca et al. (2007) and then developed in Da Fonseca et al. (2008a). We analyse the calibration property of this model and compare it with other models such as the one proposed by Barndorff-Nielsen and Stelzer (2007).
Finally, we provide some price approximations for vanilla options in the spirit of Benabid et al. (2009) that are very useful to speed up the pricing process thus leading to reasonable calibration time. Also we provide some results on Malliavin calculus that allow for efficient computation of the sensitivities for derivative products in our models.

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