Optimal Portfolio Choice with Contagion Risk and Restricted Information

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Abstract

The recent financial turmoil shows that contagion effects can threaten the financial system world wide. This paper studies the impact of contagion risk and restricted information on the portfolio decision of a single CRRA investor. In a Poisson hidden Markov model (PHMM) with two economic states and two assets, the investor infers the probability of being in the riskier contagion state from historical asset prices using filtering techniques. We find that both contagion and learning have a significant impact on the portfolio decision and, in particular, on the portfolio adjustments after jumps. With restricted information, the investor overreacts to normal, non-contagious jumps and underreacts to jumps which increase the overall level of risk in the economy. The overreaction is most pronounced in a complete market where derivatives are available, whereas the underreaction to contagion-triggering events is largest in an incomplete market. If one of the assets induces contagion more frequently, jumps in this particular asset lead to the largest overreaction, whereas the resulting portfolio adjustment mainly affects those assets which are most severely hit by contagion.

Keywords: Asset Allocation, Jumps, Contagion, Filtering

JEL: G11, G12, G13

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1 Introduction and Motivation

The notion of contagion in financial markets refers to a situation where losses in one asset, one asset class, or one country increase the risk of subsequent losses in other assets, other asset classes, or other countries. Contagion may arise due to firm-specific relations, e.g. dependency on one main customer or producer, due to macroeconomic risk factors, e.g. interest rates or business cycle variables, or due to psychological reasons, e.g. bank runs. The most recent example is the US subprime crisis which started in the financial industry in 2007 and spread to the real economy in the subsequent two years.

Contagion is characterized by several stylized facts. Firstly, contagion is not a 'one time event' in the sense that it occurs, causes immediate losses in several assets, but has no long-lasting impact otherwise. It rather describes a period of increased risk in the market, where the probability of severe losses in assets is significantly larger than in 'normal' times. Secondly, the economy does not enter the contagion state without any direct impact on asset prices, but there is a loss in one of the assets at the same time. Intuitively, there may be very bad news about company (or country) A, which make its asset price drop and, at the same time, lead to a period of higher risk in the whole economy. Alternatively, one can think of an external event (like a natural disaster in one country) that increases the overall level of risk and, at the same time, leads to a large loss in one of the assets.[1] Thirdly, it is hard for an investor to recognize contagion immediately when it occurs, i.e. to distinguish between 'normal' losses and losses that signal a change into the risky contagion state. Instead, investors need some time and some more observations of asset prices to get an idea of the true level of risk in the economy.

The main focus of our paper is how the risk of contagion changes the dynamic behavior of a CRRA investor. We solve for his optimal portfolio both in a complete and in an incomplete market and analyze his reaction to news in the market (i.e. to downward jumps). In particular, we study how the risk of contagion and the necessity to learn about the true state change his trading behavior. Furthermore, we explain how this dynamic behavior depends on asset characteristics. In particular, we compare assets that are very likely to trigger contagion with assets that are heavily influenced by the risk of contagion.

The asset prices in our economy follow jump-diffusion processes, and we assume a negative constant jump size. We model the risk of contagion by assuming two economic states, a calm and a contagion one. The jump intensities of the assets depend on the economic state: they are low in the calm state and significantly higher in the more risky contagion

[1] In the following, we will – also for ease of notation – rely on the first interpretation and say that a jump in a stock price 'induces' contagion.

state. Whenever the economy jumps from the calm into the contagion state, there is a downward jump in one of the assets at the same time. Besides these 'contagious' jumps, there are also normal downward jumps in asset prices, which are not linked to a change in the state. Furthermore, jumps back from the contagion to the calm state do not have a direct impact on prices.

These dynamics are visualized exemplarily in Figure 1. The economy is in the calm state at the beginning and enters the contagion state at time 5. At the same point in time, the price of asset A drops. Until the economy jumps back into the calm state around two periods later, the jump intensity is much higher both for asset A (which has triggered contagion) and asset B (which has not caused contagion, but is affected as well).

We focus on the impact of information and learning on the investor's behavior. In particular, we assume that the investor has restricted information and cannot observe the true state of the economy, but learns about it from the history of prices. Technically, asset price jumps are driven by a Poisson hidden Markov model (PHMM). We assume Bayesian learning, and the investor applies filtering techniques to continuously update the probability of being in the calm state. This probability drops significantly if the investor sees a jump, while it steadily moves back to one as long as no jumps are observed. The state variable 'estimated probability' thus follows a jump process with drift. Conditional on this state variable, the CRRA investor maximizes the expected utility of terminal wealth. He can trade in stocks and in the money market account. Besides this incomplete market, we also analyze the case of a complete market, in which derivatives are available.

Our main contributions are as follows. Firstly, we find that learning about the true state leads to both over- and underreaction to jumps in the market. After each observed jump, the investor lowers his estimated probability of being in the calm state. This change in the state variable causes a change in the optimal portfolio, and the investor trades to adjust his portfolio holdings. If the jump has not induced contagion, an investor with perfect information about the state of the economy would not have changed his portfolio weights. Restricted information thus leads to an overreaction to noncontagious jumps. In the aftermath of such a jump, the investor gradually adjusts his portfolio weights back to the original ones. If, on the other hand, the jump has indeed triggered contagion, the investor would adjust his portfolio much more if he observed the state of the economy perfectly. With restricted information, we thus see an underreaction to the contagion-inducing jump, since the investor waits for subsequent jumps to confirm that the economy really has entered the contagion state.

Secondly, the relative size of the portfolio adjustments caused by the first and by later jumps differs in a complete and in an incomplete market. Consider an investor who thinks to be in the calm state at the moment. In a complete market, this investor reacts most to the first jump, while his reaction to subsequent jumps is much lower. Stated differently, even a slight drop in the probability of being in the calm state leads to a large adjustment in the portfolio, while a further confirmation that the economy really has entered the contagion state has smaller effects. An investor who faces an incomplete market, on the other hand, reacts much more hesitantly to the first jump. For him, it takes several observed jumps to adjust his portfolio to the contagion state portfolio. The overreaction to noncontagious jumps is thus much more pronounced in the complete market while the underreaction to contagion-triggering events is larger in the incomplete market. To get the intuition, note that, in the complete market, the investor chooses a portfolio which allows him to optimally profit from all risk factors and their premia, while he is restricted to the packages offered by the stocks in an incomplete market. Therefore, his portfolio will be more extreme in the complete market, which automatically leads to larger changes. Additionally, with more assets available, the ability to take new information into account when setting up the portfolio increases and leads to rather large portfolio adjustments to news in the market.

Thirdly, we consider a market where some stocks are more likely to induce contagion, while other stocks are more heavily affected by contagion. The investor's reaction to stock price jumps and the trading volume in the stocks induced by jumps differ significantly across these assets. Since portfolio adjustments are triggered by updates in the probability of being in the calm state, the investor reacts way more to jumps in the stock which is more likely to induce contagion. The trading volume itself, however, is larger in the stock which is more heavily influenced by contagion, since the impact of the state on this stock is much more pronounced.

Our paper is related to the literature on continuous-time portfolio choice starting with Merton (1969, 1971). Early models with jump-diffusion processes have been developed by Aase (1984) and Jeanblanc-Picqué and Pontier (1990). Liu, Longstaff, and Pan (2003) consider a setup with jumps in stock prices and volatilities and solve for the optimal portfolio in an incomplete market. Liu and Pan (2003) and Branger, Schlag, and Schneider (2008) extend the analysis to a complete market. Wu (2003) focuses on a stochastic, but predictable investment opportunity set. Framstad, Oksendal, and Sulem (2001) set up a quite general model assuming the risky assets' prices to follow geometric Lévy processes.

Starting from these more general asset allocation studies, there are several ways to take systemic risk into account. One strand of the literature models contagion as simultaneous Poisson jumps in all stocks, e.g. Das and Uppal (2004). Kraft and Steffensen (2008) extend this approach to bond markets and default risk. Ait-Sahalia, Cacho-Diaz, and Hurd (2009) and Prokopczuk (2008) extend it to more assets and further stock market data. All these papers, however, disregard the time dimension of contagion. In particular, the probability of subsequent crashes remains the same after a joint jump.

The second strand of literature are so-called regime-switching models which were first introduced by Hamilton (1989). Ang and Bekaert (2002) apply this approach to a discrete-time asset allocation problem whereas Honda (2003) focuses on a continuous-time framework. Recent studies with different interpretation, parametrization and calibration of the regimes have been conducted by Guidolin and Timmermann (2005, 2007, 2008) and Kole, Koedijk, and Verbeek (2006). Although a regime-switching model can capture the time dimension of contagion, regime shifts are still triggered by an exogenous process (a finite-state Markov chain) and do not occur at the same time as crashes in some assets. Apart from these two main ideas of modeling contagion, other approaches have been developed. Buraschi, Porchia, and Trojani (2009), e.g., focus on the impact of stochastic correlation on an optimal portfolio and suggest contagion risk as one application of their method.

Recent papers have started to resolve some of the described shortcomings. In this respect, our paper is mostly related to Branger, Kraft, and Meinerding (2009). They focus on model risk and show that an investor modeling contagion using joint jumps can suffer severe utility losses once he is confronted with a Markov regime-switching framework. Kraft and Steffensen (2009) develop a similar model and apply it to the bond market, but focus on an complete market only. Bäuerle and Rieder (2007) set up a hidden Markov model similar to ours and employ filtering techniques as well. However, they provide closed-form solutions in several special cases whereas we concentrate on the economic implications of such a model.

Methodologically, our paper also builds on the large amount of literature on learning and incomplete information. The seminal studies of Detemple (1986) and Dothan and Feldman (1986) were among the first to employ nonlinear filtering techniques in order to deal with asset pricing and asset allocation under restricted information. They decompose these kinds of problems and show that the investor firstly solves a filtering problem, i.e. he estimates the current value of the state variable. Secondly, he decides on his optimal portfolio conditional on the estimated state variable.

The remainder of this paper is structured as follows. In Section 2, we present the exact model and the asset allocation problem faced by the investor. Section 3 gives the solution of this problem in a complete as well as in an incomplete market. In Section 4, we provide some numerical computations in order to show the impact of contagion risk on an investor's portfolio choice in more detail. Section 5 concludes. All proofs can be found in the Appendix.

2 Model Setup

2.1 The Economy

Our model is determined by the existence of two different filtered probability spaces, a 'large' one by which all the actual stock prices are generated and a 'small' one which the investor is allowed to use for the purpose of portfolio optimization. We will describe the large filtration at first, the small one will be explained in the next subsection.

The uncertainty in our economy is described by the complete filtered probability space $(\Omega, \mathcal{P}, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T^*]})$ and $\mathcal{F} = \mathcal{F}_{T^*}$. To rule out arbitrage, we assume that an equivalent martingale measure Q exists under which discounted asset prices are (local) martingales.[2]

Investors can borrow and lend using a money market account M with dynamics

$$dM(t) = M(t)\, r\, dt, \quad M(0) = 1,$$

where, for simplicity, the interest rate r is assumed to be constant. Besides, there are two stocks A and B with jump-diffusion dynamics ($i \in \{A, B\}$)

$$\frac{dS_i(t)}{S_i(t)} = \mu_i^{Z(t)} dt + \sigma_i^{Z(t)} dW_i(t) - \sum_{K\neq Z(t-)} L_i^{Z(t-),K} dN^K(t)$$

where the dynamics depend on some state variable Z. W_A and W_B are correlated Brownian motions with correlation ρ which capture normal stock price movements. Additionally, there are sudden large losses which are driven by an n-dimensional Markov chain $Z(t)$. Z is a right-continuous process with left limits (RCLL), and the associated n-dimensional counting process is $N = (N^K)_{K=1,\ldots,n}$. N^K denotes the number of transitions into state K, i.e.

$$N^K(t) = \#\Big\{s |\ s \in (0,t],\ \lim_{\tau\nearrow s} Z(\tau) \neq K,\ Z(s) = K\Big\}.$$

The loss in stock i upon a jump from state J into state K is given by $L_i^{J,K}$, and the intensity of this jump is $\lambda^{J,K}$.

We interpret the states of the economy as calm and contagion states which mainly differ with respect to the jump intensities. While the jump intensities are low in a calm state, they increase when the economy enters a contagion state. The corresponding Markov chain that allows us to capture this situation has eight states $\{cont_{A1}$, $cont_{A2}$, $cont_{B1}$, $cont_{B2}$, $calm_{A1}$, $calm_{A2}$, $calm_{B1}$, $calm_{B2}\}$ and is illustrated in Figure 2. The first subscript denotes

[2] See Delbaen and Schachermayer (1994) or Harrison and Kreps (1979) for the essential equivalence of the existence of such a measure and the absence of arbitrage.

the stock in which the most recent jump has happened. The second subscript allows us to model several subsequent jumps in the same stock which do not change the economic state: if there are several subsequent jumps in stock A in the calm state, e.g., the Markov chain jumps back and forth between $calm_{A1}$ and $calm_{A2}$.

In general, all parameters in the model can depend on the current state Z. In the following, we make the standing assumption that they coincide in all calm and in all contagion states. We will thus only refer to the two states 'calm' and 'contagion' in the following, i.e. the use of eight states is for technical reasons only. The structure of the economy is as follows: In the calm state, the intensity of a jump in stock i that does not trigger contagion is $\lambda_i^{calm,calm}$, and the corresponding loss in stock i is $L_i^{calm,calm}$. The intensity of a jump in stock i that does trigger contagion is $\lambda_i^{calm,cont}$ and the loss of stock i for such a jump is $L_i^{calm,cont}$. If the economy is in a contagion state, the intensity for a loss in stock i is $\lambda_i^{cont,cont}$, and the corresponding loss size is $L_i^{cont,cont}$. After spending some time in the contagion state, the economy will eventually jump back into the calm state. The intensity for this to happen is $\lambda^{cont,calm}$, and it is assumed that this event does not induce any losses in stocks, i.e. $L_i^{cont,calm} \equiv 0$, $i \in \{A, B\}$. Our standing assumption of equal parameters in all calm and in all contagion states implies that all calm states and all contagion states are identical in the sense that optimal portfolios and indirect utilities are the same.

Finally, we specify the drift and the risk premia of the stocks. In the general model, the drift of stock i is equal to

$$\mu_i^{Z(t)} = r + \phi_i^{Z(t)} + \sum_{K \neq Z(t-)} L_i^{Z(t-),K} \lambda^{Z(t-),K},$$

where the last term is the compensator of the jump processes. The risk premium is given by

$$\phi_i^{Z(t)} = \sigma_i^{Z(t)} \eta_i^{diff,Z(t)} + \sum_{K \neq Z(t-)} L_i^{Z(t-),K} \lambda^{Z(t-),K} \eta^{Z(t-),K},$$

where $\eta_i^{diff,K}$ is the premium for diffusive risk W_i when the economy is in state K, and $\eta^{J,K}$ is the premium for jumps from J into K. The intensity for a jump from J into K under the risk neutral measure is thus $(1 + \eta^{J,K})$ times the intensity under the physical measure.

With our definition of the Markov chain, the risk premia only depend on whether the economy is in one of the calm or in one of the contagion states. Consequently, they can be rewritten as

$$\begin{aligned} \phi_i^{calm} &= \sigma_i^{calm} \eta_i^{diff,calm} + L_i^{calm,calm} \lambda_i^{calm,calm} \eta_i^{calm,calm} + L_i^{calm,cont} \lambda_i^{calm,cont} \eta_i^{calm,cont} \\ \phi_i^{cont} &= \sigma_i^{cont} \eta_i^{diff,cont} + L_i^{cont,cont} \lambda_i^{cont,cont} \eta_i^{cont,cont}. \end{aligned}$$

Apart from stocks and the money market account, the investor might also have access to derivatives. The exposure of the derivatives to the risk factors can be calculated using Ito's lemma.

2.2 Filtering the current state

The stock price dynamics – and in particular the attractiveness of an investment into stocks – depend on the current state of the economy. In the following, we assume that the investor has restricted information and cannot observe this state, but has to infer it from past asset prices in the market. The most recent example and motivation is given by the subprime crisis where, even by the mid of 2008, some investors might still have doubted to experience the worst financial crisis for decades.

Formally, we thus deal with two different filtrations. The 'large' filtration $\mathcal{F}$ describes the true data-generating process, while the 'small' filtration $\{\mathcal{G}_t\}_{t\in[0,T^*]} \subset \{\mathcal{F}_t\}_{t\in[0,T^*]}$ captures the investor's restricted information when deciding on his optimal portfolio. $\{\mathcal{G}_t\}_{t\in[0,T^*]}$ includes the history of observed stock price jumps, but not the history of the underlying hidden Markov chain. The stock price jumps in our economy are thus determined by a Poisson hidden Markov model (PHMM).

Detemple (1986) and Dothan and Feldman (1986) show that this kind of problem can be solved in two steps. Firstly, the investor solves a filtering problem, i.e. he estimates the current value of the state variable. Secondly, he decides on his optimal portfolio conditional on the just estimated state variable.

We assume that the investor can perfectly disentangle jumps from diffusions.[3] He observes the total number of jumps $\widehat{N}_A(t)$ and $\widehat{N}_B(t)$ where

$$\widehat{N}_i(t) = N_i^{calm,calm}(t) + N_i^{calm,cont}(t) + N_i^{cont,cont}(t)$$

with the obvious meaning of the counting processes on the right hand side.[4] However, he is not able to distinguish between the three different kinds of jumps on the right hand side. Furthermore, he cannot observe jumps back from the contagion state to the calm state since these jumps do not have any impact on stock prices.

[3] This is in line with recent empirical research, see e.g. Ait-Sahalia (2004), Ait-Sahalia and Jacod (2009) or Johannes, Polson, and Stroud (2009).

[4] We will stick to this notational convention throughout the whole paper. Variables wearing a 'hat' will denote subjective parameters which the investor estimates. Variables without a 'hat' represent the true parameters in the economy.

His inference about the current state of the economy is then based on the history of $\widehat{N}_A(t)$ and $\widehat{N}_B(t)$ only. In particular, we make the simplifying assumption that he cannot learn from stock price changes which are not due to jumps, i.e. from the drift and diffusion components. The dynamics of the subjective probability $\widehat{p}_t$ of being in the calm state are

$$\begin{aligned} d\widehat{p}_t &= \Big((1-\widehat{p}_t)\lambda^{cont,calm} - \widehat{p}_t(\lambda_A^{calm,cont} + \lambda_B^{calm,cont})\Big)\, dt \\ &\quad + \widehat{p}_t \left(\frac{\lambda_A^{calm,calm}}{\widehat{\lambda}_A(\widehat{p}_t)} - 1\right)\Big(d\widehat{N}_A(t) - \widehat{\lambda}_A(\widehat{p}_t)dt\Big) \\ &\quad + \widehat{p}_t \left(\frac{\lambda_B^{calm,calm}}{\widehat{\lambda}_B(\widehat{p}_t)} - 1\right)\Big(d\widehat{N}_B(t) - \widehat{\lambda}_B(\widehat{p}_t)dt\Big), \end{aligned} \tag{1}$$

where the estimated subjective intensity of $\widehat{N}_i$ is

$$\widehat{\lambda}_i(\widehat{p}_t) = \widehat{p}_t\left(\lambda_i^{calm,calm} + \lambda_i^{calm,cont}\right) + (1-\widehat{p}_t)\lambda_i^{cont,cont}.$$

Details are given in Appendix A.1. Note that the intensity $\widehat{\lambda}_i$ depends on $\widehat{p}_t$. In the following, we will omit this dependence for the sake of readability.

The first term in brackets in Equation (1) gives the expected change of $\widehat{p}_t$. It depends on the probability of jumping back into the calm state (conditional on being in the contagion state) and the probability of leaving the calm state (conditional on being in the calm state). If there is a jump in stock i, the probability decreases from $\widehat{p}_t$ to $\widehat{p}_t\,\frac{\lambda_i^{calm,calm}}{\widehat{\lambda}_i(\widehat{p}_t)}$ and will eventually approach zero if a large number of jumps is observed. If there is no jump in the stock price, then the change in $\widehat{p}_t$ is given by the expected change and the compensators of the jump components. In line with intuition, the sum of all drift terms is positive, reflecting the fact that the subjective probability $\widehat{p}_t$ of being in the calm state will increase and eventually approach one if no jumps are observed. Note that the sum of all drift terms is indeed equal to zero iff $\widehat{p}_t = 1$.

2.3 Market Prices of Risk

We assume that the investor can observe jumps in stock prices perfectly and uses these jumps to draw his inference about $\widehat{p}_t$. For the model to be sound from an informational point of view, the investor must not be able to filter the state of the economy from any other observation than the jump processes.[5] This imposes some restrictions on the market

[5] Of course, it would be more realistic to assume that the drift rate of certain assets depends on the economic regime as well. This task is, however, more challenging and is part of our future research. A discussion of the filter theory needed in the case of time-varying drift rates can be found in Honda (2003), Haussmann and Sass (2004) and Bäuerle and Rieder (2005).

prices of jump and diffusion risk.

2.3.1 Complete Market

In a complete market, the investor can trade the stocks, the money market account and (a large enough number of) derivatives. This allows him to disentangle all (observable) risk factors. As discussed in detail in Liu and Pan (2003), he can thus abstract from the exact characteristics of the traded assets and rather focus on the exposure of his portfolio to the different risk factors.

With restricted information and learning, the investor relies on the filtration $\mathcal{G}$ when deciding on his optimal portfolio. He faces two diffusion risk factors and two jump risk factors, namely jumps in stock A and jumps in stock B. Note that he can neither distinguish the different kinds of stock price jumps nor observe jumps back from the contagion state to the calm state. With constant jump sizes, the market is thus complete relative to the filtration $\mathcal{G}$ if two derivatives are traded besides the stocks and the money market account.

In order to keep our model sound from an informational point of view, the dynamics of the four 'pure assets' which each give the investor access to exactly one of the four risk factors have to be observationally identical in the calm and the contagion state. Otherwise, the investor would be able to perfectly infer the state of the economy from these dynamics. For diffusion risk, the resulting conditions are

$$\begin{aligned} d\widehat{W}_i(t) + \widehat{\eta}_i^{diff} dt &= dW_i(t) + \eta_i^{diff,calm} dt \\ d\widehat{W}_i(t) + \widehat{\eta}_i^{diff} dt &= dW_i(t) + \eta_i^{diff,cont} dt. \end{aligned}$$

Since we do not want to focus on the estimation of diffusion risk here, we assume that

$$\widehat{\eta}_i^{diff} = \eta_i^{diff,calm} = \eta_i^{diff,cont} =: \eta_i^{diff}. \tag{2}$$

For jump risk, the conditions

$$\begin{aligned} d\widehat{N}_i(t) - \widehat{\lambda}_i \left(1 + \widehat{\eta}_i^{jump}\right) dt &= dN_i^{calm,calm}(t) - \lambda_i^{calm,calm} \left(1 + \eta_i^{calm,calm}\right) dt \\ &\quad + dN_i^{calm,cont}(t) - \lambda_i^{calm,cont} \left(1 + \eta_i^{calm,cont}\right) dt \\ d\widehat{N}_i(t) - \widehat{\lambda}_i \left(1 + \widehat{\eta}_i^{jump}\right) dt &= dN_i^{cont,cont}(t) - \lambda_i^{cont,cont} \left(1 + \eta_i^{cont,cont}\right) dt. \end{aligned}$$

simplify to

$$\begin{aligned} \widehat{\lambda}_i \left(1 + \widehat{\eta}_i^{jump}\right) &= \lambda_i^{calm,calm} \left(1 + \eta_i^{calm,calm}\right) + \lambda_i^{calm,cont} \left(1 + \eta_i^{calm,cont}\right) \\ &= \lambda_i^{cont,cont} \left(1 + \eta_i^{cont,cont}\right). \end{aligned} \tag{3}$$

This implies that the risk-neutral jump intensities are constant in our model. Since the true jump intensities depend on the current state of the economy, the market prices of jump risk do so, too. In particular, the investor earns a small risk premium in the contagion state (where the intensity is large) and a large risk premium in the calm state. Furthermore, the subjective market prices of jump risk, $\widehat{\eta}_i^{jump}$, depend on the state variable $\widehat{p}_t$.

2.3.2 Incomplete Market

In an incomplete market the investor can only trade the money market account and the two stocks. He thus cannot choose the exposure to the different risk factors separately, but is restricted to the package of risk factors as offered by the stocks.

Again, we have to choose the parameters in the calm and the contagion state in such a way that the investor cannot perfectly infer the true state from observing the stock prices. This implies that the jump sizes for the different kinds of jumps must be equal, i.e. $L_i^{calm,calm} = L_i^{calm,cont} = L_i^{cont,cont}$. Furthermore, the diffusion volatilities need to be equal since second moments of a diffusion can be estimated perfectly:

$$\widehat{\sigma}_i = \sigma_i^{calm} = \sigma_i^{cont} =: \sigma_i.$$

Additionally, we equate the terms $\mu_i^{Z(t)} - r$ and $\widehat{\mu}_i - r$:

$$\begin{aligned}
& \sigma_i \widehat{\eta}_i^{diff} + L_i \widehat{\lambda}_i (1 + \widehat{\eta}_i^{jump}) \\
= \quad & \sigma_i \eta_i^{diff,cont} + L_i \lambda_i^{cont,cont} (1 + \eta_i^{cont,cont}) \\
= \quad & \sigma_i \eta_i^{diff,calm} + L_i \lambda_i^{calm,calm} (1 + \eta_i^{calm,calm}) + L_i \lambda_i^{calm,cont} (1 + \eta_i^{calm,cont}) \\
=: \quad & \mu_i - r.
\end{aligned}$$

There is no unique solution to this system of equations. We pick the respective solution of Equations (2) and (3) in the complete market, which ensures that the analysis in the complete and the incomplete market is comparable.

2.4 The Investor

We consider an investor with CRRA-utility $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ where $\gamma > 0$ denotes his relative risk aversion. The planning horizon is denoted by $T < T^*$. The investor maximizes expected utility from terminal wealth X_T only. His indirect utility at time t conditional on the state variable $\widehat{p}_t$ and a current wealth level of X_t is defined as

$$G(t, X_t, \widehat{p}_t) = \max_{\Pi \in \mathcal{A}(t, \widehat{p}_t)} \left\{ E\left[u(X_T) | \widehat{p}_t \right] \right\},$$

where $\mathcal{A}(t, \widehat{p}_t)$ denotes the set of all admissible trading strategies. This set depends on whether the economy is complete or incomplete.

To assess the attractiveness of the investment opportunities, we rely on the certainty equivalent return (CER):

$$G(t, X_t, \widehat{p}_t) = \frac{\left(X_t e^{CER(t,\widehat{p}_t)(T-t)}\right)^{1-\gamma}}{1-\gamma}.$$

It gives the deterministic annual return on wealth that would result in the same indirect utility as the optimal investment into risky assets.

3 Asset Allocation

3.1 Complete Market

In a complete market, the investor can use derivatives to disentangle the different risk factors. As has been pointed out e.g. by Liu and Pan (2003), this means that his decision variables are the factor exposures. In our model, the investor chooses the optimal vector $(\theta_A^{diff}, \theta_B^{diff}, \theta_A^{jump}, \theta_B^{jump})$ of exposures to the four observable risk factors. The (subjective) budget equation is

$$\begin{aligned} \frac{dX_t}{X_t} &= rdt + \theta_A^{diff}(t, \widehat{p}_t) \cdot (d\widehat{W}_A(t) + \widehat{\eta}_A^{diff} dt) + \theta_B^{diff}(t, \widehat{p}_t) \cdot (d\widehat{W}_B(t) + \widehat{\eta}_B^{diff} dt) \\ &\quad + \theta_A^{jump}(t, \widehat{p}_t) \left[d\widehat{N}_A(t) - \widehat{\lambda}_A(\widehat{p}_t)dt - \widehat{\eta}_A^{jump}(\widehat{p}_t)\widehat{\lambda}_A(\widehat{p}_t)dt \right] \\ &\quad + \theta_B^{jump}(t, \widehat{p}_t) \left[d\widehat{N}_B(t) - \widehat{\lambda}_B(\widehat{p}_t)dt - \widehat{\eta}_B^{jump}(\widehat{p}_t)\widehat{\lambda}_B(\widehat{p}_t)dt \right]. \end{aligned} \tag{4}$$

Following the usual separation approach, the indirect utility function can be written as $G(t, X_t, \widehat{p}_t) = \frac{X_t^{1-\gamma}}{1-\gamma} f(t, \widehat{p}_t)$. The function f and the optimal exposures are given in the next proposition. Note that the θ_i^{jump} are functions of t and $\widehat{p}_t$ and that $\widehat{\eta}_i^{jump}$, $\widehat{\lambda}_i$ and $\widehat{p}_i^+$ depend on $\widehat{p}_t$. In the following, we will omit this dependence for the sake of readability.

Proposition 3.1 (Complete Market) *In a complete market, the optimal exposures to diffusion risk are*

$$\theta_A^{diff} = \frac{\widehat{\eta}_A^{diff} - \rho\,\widehat{\eta}_B^{diff}}{\gamma(1-\rho^2)}, \qquad \theta_B^{diff} = \frac{\widehat{\eta}_B^{diff} - \rho\,\widehat{\eta}_A^{diff}}{\gamma(1-\rho^2)}.$$

The optimal exposures to jump risk and the indirect utility function solve

$$\begin{aligned} & f_t(t,\widehat{p}_t) + f(t,\widehat{p}_t)\cdot(\mathcal{D}+\mathcal{E}) + f_p(t,\widehat{p}_t)\cdot\mathcal{B} \\ +\left(1+\theta_A^{jump}\right)^{1-\gamma}\widehat{\lambda}_A f(t,\widehat{p}_A^{+}) + \left(1+\theta_B^{jump}\right)^{1-\gamma}\widehat{\lambda}_B f(t,\widehat{p}_B^{+}) &= 0 \end{aligned} \tag{5}$$

$$-f(t,\widehat{p}_t)\cdot(1+\widehat{\eta}_A^{jump}) + f(t,\widehat{p}_A^{+})\cdot\left(1+\theta_A^{jump}\right)^{-\gamma} = 0 \tag{6}$$

$$-f(t,\widehat{p}_t)\cdot(1+\widehat{\eta}_B^{jump}) + f(t,\widehat{p}_B^{+})\cdot\left(1+\theta_B^{jump}\right)^{-\gamma} = 0 \tag{7}$$

where

$$\begin{aligned} \mathcal{B} &= (1-\widehat{p}_t)\lambda^{cont,calm} + \left(\widehat{\lambda}_A - \lambda_A^{calm,calm} - \lambda_A^{calm,cont} + \widehat{\lambda}_B - \lambda_B^{calm,calm} - \lambda_B^{calm,cont}\right)\widehat{p}_t \\ \mathcal{D} &= 0.5\frac{1-\gamma}{\gamma(1-\rho^2)}\left((\widehat{\eta}_A^{diff})^2 + (\widehat{\eta}_B^{diff})^2 - 2\rho\widehat{\eta}_A^{diff}\widehat{\eta}_B^{diff}\right) \\ \mathcal{E} &= -\left[\theta_A^{jump}\left(1+\widehat{\eta}_A^{jump}\right)(1-\gamma)+1\right]\widehat{\lambda}_A - \left[\theta_B^{jump}\left(1+\widehat{\eta}_B^{jump}\right)(1-\gamma)+1\right]\widehat{\lambda}_B \end{aligned}$$

and where $\widehat{p}_i^{+} = \frac{\lambda_i^{calm,calm}}{\widehat{\lambda}_i}\cdot\widehat{p}_t$ *denotes the updated probability after a jump in stock* i*.*

The proof is given in Appendix A.2. Equations (5)-(7) form a system of partial differential algebraic equations (PDAEs). We solve this system numerically using an explicit finite difference scheme.[6]

The optimal exposures can be decomposed into a speculative demand and a hedging demand. The optimal demand for diffusion risk is purely speculative. It does not depend on $\widehat{p}_t$ since the diffusion parameters in our model do not depend on the state of the economy. The optimal demand for jump risk, on the other hand, contains a non-zero hedging demand, since the jump parameters depend on $\widehat{p}_t$. Equations (6) and (7) can be rewritten as

$$\begin{aligned} \theta_i^{jump} &= (1+\widehat{\eta}_i^{jump})^{-\frac{1}{\gamma}}\left(\frac{f(t,\widehat{p}_t)}{f(t,\widehat{p}_i^{+})}\right)^{-\frac{1}{\gamma}} - 1 \\ &= \underbrace{(1+\widehat{\eta}_i^{jump})^{-\frac{1}{\gamma}} - 1}_{\text{speculative demand}} + \underbrace{(1+\widehat{\eta}_i^{jump})^{-\frac{1}{\gamma}}\left(\left(\frac{f(t,\widehat{p}_t)}{f(t,\widehat{p}_i^{+})}\right)^{-\frac{1}{\gamma}} - 1\right)}_{\text{hedging demand}}. \end{aligned}$$

The speculative demand equals the total demand for jump risk under the assumption of a constant investment opportunity set. It would be the optimal demand if the market

[6] Since the PDAEs are of first order, we employ upwind techniques in order to circumvent problems involving characteristic manifolds. For a detailed discussion of characteristic manifolds for hyperbolic or first-order partial differential equations and the associated numerical challenges, we refer the reader to Morton and Mayers (2005).

prices of jump risk were held constant at the current subjective levels $\widehat{\eta}_i^{jump}$. The second term is the hedging demand. A jump in stock i lowers the subjective probability $\widehat{p}_t$ of being in the calm state and influences the investor's indirect utility. If the (subjective) investment opportunities worsen, an investor with $\gamma > 1$ has a positive hedging demand for jump risk. Investment opportunities change again if the economy jumps back from the contagion state to the calm state. However, these jumps are neither observable nor tradable.

3.2 Incomplete Market

In the incomplete market, the investor can only choose the weights π_A and π_B of the two stocks in his portfolio. Since he cannot trade any other assets, he cannot implement the optimal factor exposures derived in the former section perfectly and incurs a utility loss. The budget restriction is

$$\frac{dX(t)}{X(t)} = \pi_A(t,\widehat{p}_t)\frac{dS_A(t)}{S_A(t)} + \pi_B(t,\widehat{p}_t)\frac{dS_B(t)}{S_B(t)} + [1 - \pi_A(t,\widehat{p}_t) - \pi_B(t,\widehat{p}_t)]\, r dt.$$

We define the function f by $G(t, X_t, \widehat{p}_t) = \frac{X_t^{1-\gamma}}{1-\gamma} f(t,\widehat{p}_t)$ once again and obtain the following proposition. Again, we omit the dependence of π_i and $\widehat{\mu}_i$ on its arguments for the sake of readability.

Proposition 3.2 (Incomplete Market) *In an incomplete market where the investor can only trade the two stocks and the money market account, the optimal portfolio weights and the indirect utility function solve*

$$\begin{aligned} f_t(t,\widehat{p}_t) + f(t,\widehat{p}_t)\cdot\Big[(1-\gamma)\cdot\mathcal{A}^* - 0.5\gamma(1-\gamma)\cdot\mathcal{C}^* - \widehat{\lambda}_A - \widehat{\lambda}_B\Big] + f_p(t,\widehat{p}_t)\cdot\mathcal{B}& \\ + \left[(1-\pi_A L_A)^{1-\gamma}\cdot f(t,\widehat{p}_A^+)\right]\widehat{\lambda}_A + \left[(1-\pi_B L_B)^{1-\gamma}\cdot f(t,\widehat{p}_B^+)\right]\widehat{\lambda}_B &= 0 \quad (8) \end{aligned}$$

$$\begin{aligned} f(t,\widehat{p}_t)\cdot(\widehat{\mu}_A - r) - \gamma\pi_B\rho\widehat{\sigma}_A\widehat{\sigma}_B\cdot f(t,\widehat{p}_t) - \gamma\widehat{\sigma}_A^2\pi_A\cdot f(t,\widehat{p}_t)& \\ - L_A\cdot(1-\pi_A L_A)^{-\gamma}\cdot f(t,\widehat{p}_A^+)\cdot\widehat{\lambda}_A &= 0 \quad (9) \\ f(t,\widehat{p}_t)\cdot(\widehat{\mu}_B - r) - \gamma\pi_A\rho\widehat{\sigma}_A\widehat{\sigma}_B\cdot f(t,\widehat{p}_t) - \gamma\widehat{\sigma}_B^2\pi_B\cdot f(t,\widehat{p}_t)& \\ - L_B\cdot(1-\pi_B L_B)^{-\gamma}\cdot f(t,\widehat{p}_B^+)\cdot\widehat{\lambda}_B &= 0 \quad (10) \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}^* &= r + \pi_A\cdot(\widehat{\mu}_A - r) + \pi_B\cdot(\widehat{\mu}_B - r) \\ \mathcal{B} &= (1-\widehat{p}_t)\lambda^{cont,calm} + \left(\widehat{\lambda}_A - \lambda_A^{calm,calm} - \lambda_A^{calm,cont} + \widehat{\lambda}_B - \lambda_B^{calm,calm} - \lambda_B^{calm,cont}\right)\widehat{p}_t \\ \mathcal{C}^* &= \pi_A^2\widehat{\sigma}_A^2 + \pi_B^2\widehat{\sigma}_B^2 + 2\pi_A\pi_B\rho\widehat{\sigma}_A\widehat{\sigma}_B \end{aligned}$$

and where $\widehat{p}_i^+ = \frac{\lambda_i^{calm,calm}}{\widehat{\lambda}_i} \cdot \widehat{p}_t$ *denotes the updated probability after a jump in stock* i*.*

The proof can be found in Appendix A.3. As in the complete market, equations (8)-(10) form a system of partial differential algebraic equations which can be solved numerically using a finite difference scheme.

Again, the indirect utility function and the optimal portfolio weights π_i depend on the state variable $\widehat{p}_t$. Since $\widehat{p}_t$ evolves stochastically following a jump process with drift, the optimal portfolio weights do so, too. As in the complete market, the optimal portfolio lies somewhere between the calm and contagion state portfolio. If no jump is observed, it is gradually adjusted towards the optimal portfolio in the calm state. If a jump happens, it is adjusted by a discrete amount towards the optimal portfolio in the contagion state.

4 Numerical Results

4.1 Parametrization and Calibration

We consider a CRRA investor with a relative risk aversion of $\gamma = 3$ and a planning horizon of 5 years. The riskless interest rate is set to $r = 0.01$. In the benchmark case, the stocks are assumed to follow identical stochastic processes. Furthermore, we assume that only the jump intensities and market prices for jump risk differ between the calm and the contagion state, while the diffusion parameters and the loss size do not depend on the current state.

We choose some representative parameters for our model, but do not provide a full estimation or calibration of the model. In particular, we roughly rely on the study of Eraker, Johannes, and Polson (2003), who estimate the parameters of a jump-diffusion model under the true physical measure from S&P500 and Nasdaq 100 index returns, and on the results of Broadie, Chernov, and Johannes (2007), who calibrate the model to option prices, based on the parameter estimates of Eraker, Johannes, and Polson (2003).

The diffusion volatility σ is set to 0.15, and the two Wiener processes are correlated with $\rho = 0.5$. The market price of diffusion risk η_i^{diff} is set to 0.35 which gives a constant diffusion risk premium of 0.0525. The average jump intensity equals 0.62. The difference between the jump intensities in the calm and the contagion state is captured by $\xi_i \geq 1$:

$$\lambda_i^{cont,cont} = \xi_i \left(\lambda_i^{calm,calm} + \lambda_i^{calm,cont}\right), \quad i \in \{A, B\},$$

where we set ξ_i equal to 5 in the benchmark case. The jump intensity of both stocks is thus multiplied by 5 as soon as the economy enters the contagion state. The conditional probability that a loss in one of the stocks actually triggers contagion is given by α_i:

$$\lambda_i^{calm,cont} = \alpha_i \left(\lambda_i^{calm,calm} + \lambda_i^{calm,cont}\right), \quad i \in \{A, B\}$$

For the benchmark case, we set $\alpha_i = 0.2$. On average, every fifth jump in the calm state triggers contagion. The average time the economy stays in the contagion state is governed by ψ:

$$\lambda^{cont,calm} = \psi \left(\lambda_A^{cont,cont} + \lambda_B^{cont,cont}\right).$$

We start with $\psi = 0.25$. With $\lambda_i^{calm,calm} = 0.32$, the other jump intensities then equal $\lambda_i^{calm,cont} = 0.08$, $\lambda_i^{cont,cont} = 2$ and $\lambda^{cont,calm} = 1$. Consequently, the average time the economy stays in the calm state is 6.25 years, and the average duration of the contagion regime is 1 year. The constant jump size is assumed to be -4%, i.e. the loss size L equals 4%.

In the calm state, we set the total jump risk premium equal to 0.08 and the premium for jumps that do not trigger contagion equal to 0.0256. The market prices of risk for all other jump events then follow from the restrictions in Section 2.3. In particular, the jump risk premium in the contagion state is 0.016 and thus significantly lower than in the calm state. The first column of Table 1 gives an overview over our benchmark parametrization.

4.2 Optimal Portfolios in the Benchmark Case

Figure 3 gives the solution in the benchmark case. The upper row shows the demand for jump risk in the complete market as a function of the state variable $\widehat{p}_t$, where the left graph gives the total demand, while the middle and right one give the speculative demand and the hedging demand, respectively. Since both stocks have identical parameters, the demand for jump risk in stock A and stock B is identical.

The premium for (downward) jumps is positive. Consequently, the investor has a negative speculative demand for jump risk, i.e. he is willing to accept losses up to 45% of his wealth if a jump occurs in order to earn the positive risk premium. The market price of jump risk is larger in the calm than in the contagion state. Therefore, the speculative demand is way higher in absolute terms for a high probability of being in the calm state ($\widehat{p}_t$ close to one) than for $\widehat{p}_t$ close to zero. The diffusion exposures (not shown in the figure) equal 0.0778 in both states.

With different jump intensities and different market prices of risk, investment opportunities differ between the calm and contagion state. As can be seen from the middle graph in the lower row, the certainty equivalent return is larger in the calm state (for $\widehat{p}_t = 1$) than in the contagion state (for $\widehat{p}_t = 0$), which is in line with the equity risk premium being larger in the calm state as well. This difference in investment opportunities induces a hedging demand for jump risk. An investor with $\gamma > 1$ wants to get a compensation for the impairment in investment opportunities if $\widehat{p}_t$ decreases. This decrease in $\widehat{p}_t$ is induced by a jump in any of the stocks, so that there is a positive hedging demand for jumps in stock A and B. The positive hedging demand partly offsets the negative speculative demand and results in a total negative demand for jump risk that is lower than the speculative demand in absolute terms.

The hedging demand is shown in the right upper graph in Figure 3. It first increases and then decreases in $\widehat{p}_t$ and is thus a non-monotonic function of the state variable $\widehat{p}_t$. To get the intuition, note that the hedging demand for jump risk depends on the impact of a jump on the utility, which in turn depends on the impact of the jump on the state variable $\widehat{p}_t$. The lower right graph gives the updated probability $\widehat{p}_i^+$ immediately after a jump as a function of the probability $\widehat{p}_t$ before that jump. It shows that the update $\widehat{p}_t - \widehat{p}_i^+$ (the difference between the grey bisecting line and the updated probability) is largest for $\widehat{p}_t$ around 0.8. Furthermore, the impact of a change in $\widehat{p}_t$ on the CER is the larger the higher $\widehat{p}_t$ due to the convexity of the CER. In total, the hedging demand is thus largest for $\widehat{p}_t$ around 0.9.

If $\widehat{p}_t$ is equal to zero, on the other hand, the investment opportunity set cannot worsen any more, and consequently, the hedging demand vanishes. Jumps back from the contagion state to the calm state improve the investment opportunities. Since these jumps are neither observable nor tradable, however, the investor obviously cannot hedge against them.

For $\widehat{p}_t$ close to one, the hedging demand can be as large as 50% of the speculative demand in absolute terms. Modeling contagion as a separate state of the economy thus has a significant impact on the overall optimal portfolio.[7]

If the investor has to learn about the true state of the economy, his exposure depends on the estimated probability $\widehat{p}_t$ of being in the calm state and lies somewhere between the optimal exposures in the calm and in the contagion state. The non-monotonic shape of the hedging demand, however, shows that the investor does not simply interpolate linearly between the calm and the contagion state portfolio.

We now turn to the dynamic implications of learning. Assume that we start from $\widehat{p}_t = 1$.

[7] For a detailed discussion, see also Branger, Kraft, and Meinerding (2009).

The investor adjusts the probability to 0.8 after one jump and to 0.35 after two jumps, no matter which of the two stocks has jumped. The impact on the total demand for jump risk can be inferred from the upper left graph in Figure 3, where the crosses mark the updated subjective probabilities $\widehat{p}_t$ after one and two jumps. The optimal exposure to jump risk is approximately halved in reaction to each of these two jumps with a much larger absolute reaction to the first jump than to the second jump.

The investor underreacts to jumps that induce contagion and overreacts to normal jumps. If the economy really has changed to the contagion state, a fully informed investor should switch to the optimal portfolio in the contagion state in one single step. However, the reaction of an investor with restricted information is too small, and it takes several subsequent jumps for the investor to gradually build up the optimal contagion state portfolio. Furthermore, note that the reaction to the first jump (which may have induced contagion) is largest in absolute terms, while the reaction to each subsequent jump (which confirms that contagion has indeed been triggered) is much smaller. If, on the other hand, the jump has not triggered contagion, the investor overreacts to the jump event by adjusting the weights towards the contagion portfolio, while keeping them constant would have been the right decision if he was fully informed. With no jumps happening in the aftermath of the first one, he will then continuously adjust his portfolio back to the optimal portfolio in the calm state.

The optimal portfolio weights in an incomplete market are given in the lower left graph of Figure 3. Now, the investor can only trade stocks and the money market account. In the complete market, he reduced the amount of jump risk in his portfolio after a jump while holding the amount of diffusion risk constant. In the incomplete market, he can no longer disentangle the different risk factors, but will choose a position that minimizes the difference to the overall optimal exposures. This still makes him lower his position when the contagion state is more probable. The relative weight of each of the stocks is more than halved if the investor believes to be in the contagion state as compared to the calm state. Different from the complete market, however, the reaction to the first jump (which decreases $\widehat{p}_t$ from 1.0 to 0.8) is smaller than the reaction to the second jump (which decreases the probability to 0.35). As in the complete market, we see an overreaction to normal jumps which do not trigger contagion and an underreaction to contagion-inducing events. The effects are, however, diluted due to market incompleteness.

The investor is better off in a complete market than in an incomplete market. Having access to derivatives approximately doubles the certainty equivalent return. Furthermore, access to derivatives also changes the effect of jumps on the trading volume. In a complete market, the trading volume is largest in reaction to the first jump (which leads to a risk of having entered the contagion state), while in the incomplete market, the reaction to the second jump (which kind of confirms that the economy is in the contagion state) is larger. The larger reaction to the first jump in the complete market reflects that the investor is better able to react to the risk of contagion. He is thus willing to take significantly more jump risk in the more attractive calm state. The resulting larger trading volume in the complete market translates into a pronounced trading volume in derivatives, too.

4.3 Optimal Portfolios When Assets are Heterogeneous

Up to now, we have assumed identical parameters for the two stocks. Now, we turn to the more realistic situation where the assets differ. In particular, we consider a situation where stock A is affected more heavily by contagion than stock B, while stock B is more likely to induce contagion. One can think of stock A representing a developing country or a small sub-supplier depending on one main customer and of stock B as a developed country or a large company being linked to commercial partners all around the globe. In terms of the subprime crisis, the assets might (in a very stylized way) also represent the real economy on the one hand and the financial sector on the other hand.

The parameters for this case are given in the second and third column of Table 1. We reduce the multiplicator ξ_B to 2.5, i.e. the jump intensity of stock B is multiplied with 2.5 as soon as the economy enters the contagion state, whereas ξ_A remains equal to 5. The parameter α_B is increased from 0.2 to 0.5, so that now every second jump in stock B triggers contagion (as opposed to every fifth jump in stock A). The overall jump intensity in the calm state remains equal to 0.4 for both stocks. For stock B, the market prices of jump risk have to be adjusted according to the restrictions in Section 2.3, and the premium for jumps in the contagion state increases to 5.6%.

The results are given in Figure 4, where we also include the results from the benchmark case for comparison. We discuss the optimal demand in the complete market at first. The speculative demand for jumps in asset A (which suffers the most from contagion), for which the market prices of risk have not changed, is identical to the benchmark case. The speculative demand for jumps in asset B (which mainly triggers contagion) is larger in absolute terms in the contagion state due to a larger market price of risk. Since we have not changed any diffusion parameters, the optimal diffusion exposures in the complete market (not exhibited here) still equal 0.0778 in both states.

The hedging demand is again positive and reflects the compensation which the investor demands for the worsening of investment opportunities due to a jump. For $\widehat{p}_t$ close to one, a significant decrease in this probability is linked to the economy entering the contagion state. Since contagion is most likely triggered by stock B, the hedging demand in stock B is larger than the one in stock A. If the probability $\widehat{p}_t$ is sufficiently lower than 1, asset A may also represent a good hedge against the risk of contagion. Now, the investor is mainly interested in a hedge against the confirmation that the economy is indeed in the contagion state. With a higher jump intensity in the contagion state for asset A as compared to asset B, asset A is better suited to corroborate whether contagion really occurred or whether an observed stock price jump was just 'false alarm'.

The different information conveyed by jumps in asset A and asset B can also be seen in the lower right graph of Figure 4. For $\widehat{p}_t$ close to one, a jump in asset B (which is more likely to trigger contagion) induces a larger adjustment in the probability than a jump in asset A. For a small $\widehat{p}_t$, however, the reaction to a jump in asset A (which is more heavily affected by contagion) is slightly larger than that to a jump in asset B. These differences also lead to a structural difference in the hedging demand. The hedging demand for jumps in asset B is now a monotonic function of $\widehat{p}_t$. Recalling the discussion from the previous section, this confirms that asset B is a rather contagion-sensitive asset for $\widehat{p}_t$ close to one and becomes contagion-insensitive for lower values of $\widehat{p}_t$.

The dynamic implications of learning again follow from the total demand in the upper left graph in Figures 4. Starting from $\widehat{p}_t = 1$, the red and blue crosses mark the updated probabilities after one and two jumps in asset A and asset B, respectively. If a jump occurs, the investor adjusts the exposures to jumps in asset A and asset B. A jump in stock B, which is more likely to induce contagion, leads to a larger change in the probability and thus also to a larger change in the overall portfolio than a jump in stock A.

Furthermore, the optimal position in jumps in stock B hardly depends on the state variable. The dependence of the speculative demand on $\widehat{p}_t$ is nearly offset by the hedging demand. The exposure against jumps in stock A, on the other hand, declines by approximately two thirds in response to the first jump in asset B. The investor notices that jump risk 'spills over' from stock B to stock A and reacts accordingly.

These results hold for the incomplete market as well. The lower left graph of Figure 4 gives the optimal portfolio weights in the incomplete market. As in the benchmark case with identical stocks, all effects are diluted in the incomplete market. The investor cannot disentangle jump and diffusion risk here, but instead has to accept the whole 'package' of risk offered to him in the form of stocks. Since the diffusion parameters are equal across states, the contagion effects are less pronounced. Nevertheless, it again holds true that jumps in stock B trigger large adjustments of the portfolio, which however mainly take place in stock A. Stock B thus drives the economy, while stock A is the dependent one.

As in the benchmark case, the reaction to the first jump is largest in the complete market, while the investor reacts mostly to the second jump in the incomplete market. With differences between the stocks, however, trading is now mainly concentrated in the contagion-sensitive stock and its derivatives.

4.4 Diffusion versus Jump Risk

In the preceding sections, we have assumed that the stocks differ with respect to the jump parameters only, while the diffusion volatilities are identical. Now, we also allow for differences in diffusion risk. We start from the case with different jump intensities discussed in Section 4.3. In a first step, stock A is not only more severely hit by contagion, but also has a higher diffusion volatility. It is thus more risky than stock B in a double sense, and the economy becomes more risky, too. In a second step, we additionally set the correlation between the diffusion components equal to zero, thus decreasing the amount of diffusion risk in the economy. Note that the market prices of risk are kept constant.

The results are given in Figure 5, where we concentrate on the incomplete market. The left graph gives the optimal portfolio weights. It shows that the difference in diffusion parameters has a significant impact on the optimal portfolio. Increasing the volatility σ_A to 0.3 decreases the optimal demand for stock A by more than three fourths in the calm state and by more than one half in the contagion state. The optimal demand for stock B, on the other hand, increases slightly. Although we have tried several jump parameter specifications, this result cannot even approximately be generated by the jump parameters alone.

If we also set ρ equal to zero, the demand for both assets increases, with the larger impact on stock A. This increase is approximately the same for all values of $\widehat{p}_t$ and can be attributed to improved diversification.

As described in the previous sections, the effects of contagion risk and jumps on optimal portfolios are diluted if the investor faces an incomplete market and cannot trade derivatives. This trend is even more pronounced with the larger amount of diffusion risk. While the complete market factor exposures hardly change at all if the volatility σ_A doubles (not reported in the figures for the sake of brevity), the incomplete market portfolio weights flatten out. They depend almost linearly on $\widehat{p}_t$ and this dependence mostly disappears in the case where $\sigma_A = 0.3$ and $\rho = 0$. The reason is that the investor kind of averages the (constant) exposure to diffusion risk and the (convex) exposure to jump risk when facing an incomplete market. The more important diffusion risk becomes, the less pronounced the dependence of the optimal portfolio weights on the state variable $\widehat{p}_t$.

This implies that the portfolio revisions due to jumps are rather small. For stock market contagion to have an impact on an investor's portfolio choice, jump risk has to account for a significant part of the overall risk in the economy. The utility losses due to market incompleteness follow from the certainty equivalent returns given in the right graph of Figure 5. In all cases, the utility loss is largest in the calm state. If the relative importance of contagion (and thus of jumps) vanishes, the additional utility from having access to the derivatives market nearly vanishes as well. Obviously, the derivatives market plays a major role in establishing the correct jump risk exposures whereas its significance decreases if diffusion risk prevails.

Reducing the correlation ρ from 0.5 to 0 offsets the effects of an increasing volatility as figure 5 shows. It lowers the overall diffusion risk in the economy and the jump (and contagion) risk again becomes prevalent. Altogether, these findings indicate that our dynamic notion of contagion can have a crucial impact on an investor's portfolio choice. However, this impact naturally depends on the relative importance of jump risk as compared to diffusion risk.

4.5 Robustness Checks

We first analyze the impact of jump risk premia. In the benchmark case given in Table 1, they are 8% in the calm state and 1.6% in the contagion state and depend on the jump intensities as well as on the market prices of jump risk. Due to the restrictions on the risk premia derived in Section 2.3, we cannot change only one of these parameters. We thus consider three sets of scenarios. In the first step, we adjust the market prices of jump risk such as to lower the jump risk premium in the calm and the contagion state by the same amount (up to 4% each while holding all other variables constant). The jump risk premium in the contagion state can become negative in this case. In a second step, we additionally adjust the wedge ξ_i between the jump intensities in the calm and in the contagion state in such a way that the ratio between the risk premia equals that in the benchmark case, i.e. the jump risk premium in the calm state stays five times as large as the (now positive) jump risk premium in the contagion state. In a third step, we adjust ξ_i even further to get a jump risk premium of 4% in the calm state and of 1.6% in the contagion state. In all cases, the results remain qualitatively the same both in the complete and in the incomplete market, while the numbers are of course rescaled reflecting the changed wedge between the calm and contagion state.

We have also varied the loss size L_i and considered levels of 3% and 5%. Again, we have simultaneously adjusted the market prices of risk to keep the risk premia for diffusion and for jump risk constant. Neither the optimal portfolios nor the certainty equivalent returns change qualitatively.

Additionally, we have varied the investment horizon and the relative risk aversion. Our numerical results hardly depend on the investment horizon at all. However, the certainty equivalent returns flatten out for investment horizons of 20 years and longer: in the long run, it does not matter whether the world is in the calm or contagion state today.[8] Changing the investor's relative risk aversion between values of 1.5 and 5 does not yield any further insights as well. Of course, all numbers are rescaled: a more risk-averse investor will take less risk and incur additional utility losses therefore.

The average duration of the contagion regime is governed by the parameter ψ. Up to now, we have assumed $\psi = 0.25$. Given that the jump intensities of both stocks in the contagion state equal 2.00, this implies an average duration of the contagion regime of one year. We have varied ψ from 0.05 to 1.00 and find that, interestingly, the duration of contagion has only marginal effects. The numerical results are hardly changed at all implying that it is not the duration of contagion which matters. Instead, the mere fact that there is a contagion state which lasts for more than a second already drives our main findings.

The last check concerns the fact that, up to now, short-selling has been excluded intrinsically by choosing the risk premia appropriately. The optimal portfolio contained only long positions in all the assets (in the incomplete market) and only short positions in jump risk (in the complete market). Of course, we can set the market prices of risk in such a way that the investor optimally implements the opposite positions in his portfolio. This happens if the overall equity premium becomes negative (in the incomplete market) or if the premium on one of the risk factors becomes negative (in the complete market). We have analyzed our model with lower market prices of risk. However, as expected, this does not generate any qualitative changes, but only results in parallel shifts in all the figures.

5 Conclusion

The existence of contagion effects in financial markets is one of the main insights of the recent financial turmoil. In a Poisson hidden Markov model (PHMM) with two economic states, we solve for the optimal asset allocation of a CRRA investor in a complete market as well as in an incomplete market. We explicitly take into account that the investor

[8] or, as John Maynard Keynes stated it: 'In the long run, we are all dead.'

cannot observe the true state of the economy, but has to infer the probability of being in the calm or contagion state from the jumps observed in the market.

Our results show that the risk of stock market contagion and restricted information can have a substantial impact on an investor's optimal portfolio choice and on the trading volume. Since the investor only learns gradually about the economy having entered the contagion state, he adjusts his portfolio towards the optimal contagion state portfolio gradually, too, thereby underreacting to jumps that induce contagion. On the other hand, he overreacts to jumps that do not trigger contagion. If no further jumps occur and he learns about his overreaction over time, he continuously adjusts his portfolio back to the optimal calm state portfolio. The additional trading caused by normal jumps is particularly pronounced in a complete market where derivatives are available, while the trading volume is dampened in an incomplete market.

Allowing for differences between the stocks, we find that the trading volume is largest in those stocks that are more severely hit by contagion. On the other hand, it is rather small in stocks that trigger contagion, and jumps in these stocks mainly cause trading in other stocks. We also find that the hedging demand can become rather large. In particular, the hedging demand can be a nonmonotonic function of the probability of being in the calm state.

There are several directions for future research. Firstly, one could apply more sophisticated filtering techniques so that the investor can learn about the economic state from the stocks' drift rates as well as from their jump histories. Furthermore, the numerical results show that the assumptions about the market prices of risk have a significant impact on the optimal portfolio choice. It would thus be interesting to compute these market prices endogenously in a general equilibrium setup. In particular, we are interested in the equilibrium market price of contagion risk, and in the differences between the prices of stocks that mainly induce contagion and stocks that are more severely affected by contagion.

A Proofs

A.1 Filtering

For the moment, we denote the true probability of being in state K of the Markov chain at time t by $\zeta_t(K) = P\left[Z(t) = K\right]$ and the investor's subjective probability by $\widehat{\zeta}_t(K) = E[\zeta_t(K)|\mathcal{G}_t]$. Applying the Wonham filter[9] reveals that the subjective probability equals

$$\begin{aligned}
\widehat{\zeta}_t(K) \quad &= \quad \zeta_0(K) + \int_0^t \left(\sum_{J \neq K} \widehat{\zeta}_s(J)\lambda^{JK} - \widehat{\zeta}_s(K) \sum_{J \neq K} \lambda^{KJ} \right) ds \\
&\quad + \int_0^t F_{A,s}(K) \left(d\widehat{N}_A(s) - \sum_{all\ obs.\ jumps\ in\ A} \widehat{\zeta}_s(J)\lambda^{JL} ds \right) \\
&\quad + \int_0^t F_{B,s}(K) \left(d\widehat{N}_B(s) - \sum_{all\ obs.\ jumps\ in\ B} \widehat{\zeta}_s(J)\lambda^{JL} ds \right)
\end{aligned}$$

with

$$F_{i,t}(K) = -\widehat{\zeta}_{t-}(K) + \frac{\sum_{all\ obs.\ jumps\ in\ stock\ i\ into\ state\ K} \widehat{\zeta}_{t-}(J)\lambda^{JK}}{\sum_{all\ obs.\ jumps\ in\ stock\ i} \widehat{\zeta}_{t-}(J)\lambda^{JL}}.$$

where the Poisson processes $\widehat{N}_i$ and the sums in the second and third line cover all those jumps which trigger a loss in the respective stock and are observable for the investor, i.e. all jumps except those from the contagion to the calm state which by assumption do not cause any losses in either of the two stocks.

The subjective probability of being in the calm state, $\widehat{p}_t$, can now be extracted as $\widehat{p}_t = 2(\widehat{\zeta}_t(calm_{A1}) + \widehat{\zeta}_t(calm_{B1}))$. If one plugs in all notational simplifications from subsection 2.1 and makes use of the fact that $\widehat{p}_t = 2(\widehat{\zeta}_t(calm_{Ai}) + \widehat{\zeta}_t(calm_{Bj}))$ holds for all combinations of $i, j \in \{1, 2\}$, one ends up with

$$\begin{aligned}
d\widehat{p}_t \quad &= \quad \left((1 - \widehat{p}_t)\lambda^{cont,calm} - \widehat{p}_t(\lambda_A^{calm,cont} + \lambda_B^{calm,cont}) \right) dt \\
&\quad + \left(\frac{\widehat{p}_t \lambda_A^{calm,calm}}{\widehat{\lambda}_A(\widehat{p}_t)} - \widehat{p}_t \right) \left(d\widehat{N}_A(t) - \widehat{\lambda}_A(\widehat{p}_t)dt \right) \\
&\quad + \left(\frac{\widehat{p}_t \lambda_B^{calm,calm}}{\widehat{\lambda}_B(\widehat{p}_t)} - \widehat{p}_t \right) \left(d\widehat{N}_B(t) - \widehat{\lambda}_B(\widehat{p}_t)dt \right)
\end{aligned}$$

[9]For a detailed discussion of this filter see e.g. Brémaud (1981), pp. 94ff. A thorough mathematical analysis of the Wonham filter in a model similar to ours can also be found in the paper of Bäuerle and Rieder (2007).

where the (estimated subjective) intensity of $\widehat{N}_i$ is

$$\widehat{\lambda}_i(\widehat{p}_t) = \widehat{p}_t \left(\lambda_i^{calm,calm} + \lambda_i^{calm,cont}\right) + (1-\widehat{p}_t)\lambda_i^{cont,cont}.$$

A.2 Complete Market

We denote the indirect utility function for a current wealth level X_t and a current state variable (subjective probability of the calm state) $\widehat{p}_t$ by $G(t, X_t, \widehat{p}_t)$ and its partial derivatives by G_t, G_X, G_{XX} and G_p. Taking the drift of the budget restriction from (4), the drift of the filter equation from (1) and the squared volatility of the budget restriction from (4), the function G must solve the following Hamilton-Jacobi-Bellman equation:

$$\begin{aligned}&\max\left\{G_t + G_X \cdot \mathcal{A}X_t + G_p \cdot \mathcal{B} + 0.5 G_{XX} \cdot \mathcal{C}X_t^2\right.\\ &\left. + \left[G\left(t, X_A^+(t,\widehat{p}_t), \widehat{p}_A^+\right) - G(t, X_t, \widehat{p}_t)\right]\widehat{\lambda}_A + \left[G\left(t, X_B^+(t,\widehat{p}_t), \widehat{p}_B^+\right) - G(t, X, \widehat{p}_t)\right]\widehat{\lambda}_B\right\} = 0\end{aligned}$$

with

$$\begin{aligned}\mathcal{A} &= \theta_A^{diff}\widehat{\eta}_A^{diff} + \theta_B^{diff}\widehat{\eta}_B^{diff} - \theta_A^{jump}\cdot(\widehat{\lambda}_A + \widehat{\eta}_A^{jump}\widehat{\lambda}_A) - \theta_B^{jump}\cdot(\widehat{\lambda}_B + \widehat{\eta}_B^{jump}\widehat{\lambda}_B)\\ \mathcal{B} &= (1-\widehat{p}_t)\lambda^{cont,calm} - \widehat{p}_t(\lambda_A^{calm,cont} + \lambda_B^{calm,cont}) + \left(\widehat{\lambda}_A - \lambda_A^{calm,calm} + \widehat{\lambda}_B - \lambda_B^{calm,calm}\right)\cdot\widehat{p}_t\\ \mathcal{C} &= (\theta_A^{diff})^2 + (\theta_B^{diff})^2 + 2\rho\theta_A^{diff}\theta_B^{diff}.\end{aligned}$$

Note that the filter equation (1) contains only drift terms and pure jump processes so that all second order partial derivatives involving p vanish and do not show up in the Hamilton-Jacobi-Bellman equation.[10] The variables X_i^+ and $\widehat{p}_i^+$ give the wealth and the subjective probability immediately after a jump in stock i and are defined as

$$\begin{aligned}X_i^+(\widehat{p}_t) &= \left(1 + \theta_i^{jump}(t,\widehat{p}_t)\right)\cdot X_t\\ \widehat{p}_i^+(\widehat{p}_t) &= \widehat{p}_t\frac{\lambda_i^{calm,calm}}{\widehat{p}_t\left(\lambda_i^{calm,calm} + \lambda_i^{calm,cont}\right) + (1-\widehat{p}_t)\lambda_i^{cont,cont}} = \widehat{p}_t\frac{\lambda_i^{calm,calm}}{\widehat{\lambda}_i}.\end{aligned}$$

Although θ_i^{jump}, $\widehat{\eta}_i^{jump}$, $\widehat{p}_i^+$ and $\widehat{\lambda}_{i,t}$ are functions of $\widehat{p}_t$ (and t), we will omit this dependence in the notation for the sake of readability. We now follow the usual separation approach and plug in the guess $G(t, X_t, \widehat{p}_t) = \frac{X_t^{1-\gamma}}{1-\gamma}f(t,\widehat{p}_t)$ so that wealth factors out:

$$\begin{aligned}\max\{&f_t(t,\widehat{p}_t) + (1-\gamma)\cdot f(t,\widehat{p}_t)\cdot\mathcal{A} + f_p(t,\widehat{p}_t)\cdot\mathcal{B} - 0.5\gamma(1-\gamma)\cdot f(t,\widehat{p}_t)\cdot\mathcal{C}\\ &+ \left[\left(1+\theta_A^{jump}\right)^{1-\gamma}\cdot f(t,\widehat{p}_A^+) - f(t,\widehat{p}_t)\right]\widehat{\lambda}_A\\ &+ \left[\left(1+\theta_B^{jump}\right)^{1-\gamma}\cdot f(t,\widehat{p}_B^+) - f(t,\widehat{p}_t)\right]\widehat{\lambda}_B\Big\} = 0. \qquad (11)\end{aligned}$$

[10] Unfortunately and unintuitively, this makes the numerical computation via finite differences a bit more complicated since we are now dealing with a first-order partial differential equation and have to take its characteristic manifolds into account.

f_t and f_p denote partial derivatives again here. Deriving (11) with respect to the four exposures $\Theta = [\theta_A^{diff}, \theta_B^{diff}, \theta_A^{jump}, \theta_B^{jump}]$ gives the following first-order conditions:

$$\begin{aligned}
\widehat{\eta}_A^{diff} - \gamma\rho\theta_B^{diff} - \gamma\theta_A^{diff} &= 0 \qquad (12)\\
\widehat{\eta}_B^{diff} - \gamma\rho\theta_A^{diff} - \gamma\theta_B^{diff} &= 0 \qquad (13)\\
-f(t,\widehat{p}_t)\cdot(\widehat{\lambda}_A + \widehat{\eta}_A^{jump}\widehat{\lambda}_A) + \widehat{\lambda}_A \cdot f(t,\widehat{p}_A^+)\cdot\left(1+\theta_A^{jump}\right)^{-\gamma} &= 0 \qquad (14)\\
-f(t,\widehat{p}_t)\cdot(\widehat{\lambda}_B + \widehat{\eta}_B^{jump}\widehat{\lambda}_B) + \widehat{\lambda}_B \cdot f(t,\widehat{p}_B^+)\cdot\left(1+\theta_B^{jump}\right)^{-\gamma} &= 0. \qquad (15)
\end{aligned}$$

θ_A^{diff} and θ_B^{diff} can be computed from equations (12) and (13). Plugging the expressions

$$\theta_A^{diff} = \frac{\widehat{\eta}_A^{diff} - \rho\widehat{\eta}_B^{diff}}{\gamma(1-\rho^2)}, \qquad \theta_B^{diff} = \frac{\widehat{\eta}_B^{diff} - \rho\widehat{\eta}_A^{diff}}{\gamma(1-\rho^2)}.$$

into equation (11), exploiting the fact that

$$\begin{aligned}
&(1-\gamma)\left(\theta_A^{diff}\widehat{\eta}_A^{diff} + \theta_B^{diff}\widehat{\eta}_B^{diff}\right) - 0.5\gamma(1-\gamma)\left((\theta_A^{diff})^2 + (\theta_B^{diff})^2 + 2\rho\theta_A^{diff}\theta_B^{diff}\right)\\
= &\underbrace{0.5\frac{(1-\gamma)}{\gamma(1-\rho^2)}\left((\widehat{\eta}_A^{diff})^2 + (\widehat{\eta}_B^{diff})^2 - 2\rho\widehat{\eta}_A^{diff}\widehat{\eta}_B^{diff}\right)}_{=:\mathcal{D}}
\end{aligned}$$

and simplifying, one arrives at the partial differential algebraic system

$$\begin{aligned}
f_t(t,\widehat{p}_t) + f(t,\widehat{p}_t)\cdot(\mathcal{D}+\mathcal{E}) + f_p(t,\widehat{p}_t)\cdot\mathcal{B}&\\
+\left(1+\theta_A^{jump}\right)^{1-\gamma}\widehat{\lambda}_A f(t,\widehat{p}_A^+) + \left(1+\theta_B^{jump}\right)^{1-\gamma}\widehat{\lambda}_B f(t,\widehat{p}_B^+) &= 0 \qquad (16)\\
-f(t,\widehat{p}_t)\cdot(1+\widehat{\eta}_A^{jump}) + f(t,\widehat{p}_A^+)\cdot\left(1+\theta_A^{jump}\right)^{-\gamma} &= 0 \qquad (17)\\
-f(t,\widehat{p}_t)\cdot(1+\widehat{\eta}_B^{jump}) + f(t,\widehat{p}_B^+)\cdot\left(1+\theta_B^{jump}\right)^{-\gamma} &= 0 \qquad (18)
\end{aligned}$$

where $\mathcal{E}$ is defined as

$$\begin{aligned}
\mathcal{E} &:= -(1-\gamma)\left(\theta_A^{jump}\cdot(\widehat{\lambda}_A + \widehat{\eta}_A^{jump}\widehat{\lambda}_A) + \theta_B^{jump}\cdot(\widehat{\lambda}_B + \widehat{\eta}_B^{jump}\widehat{\lambda}_B)\right) - \widehat{\lambda}_A - \widehat{\lambda}_B\\
&= -\left[\theta_A^{jump}\left(1+\widehat{\eta}_A^{jump}\right)(1-\gamma)+1\right]\widehat{\lambda}_A - \left[\theta_B^{jump}\left(1+\widehat{\eta}_B^{jump}\right)(1-\gamma)+1\right]\widehat{\lambda}_B.
\end{aligned}$$

This completes the proof.

A.3 Incomplete Market

We denote the indirect utility function for a current wealth level X_t and a current state variable (subjective probability of the calm state) $\widehat{p}_t$ by $G(t, X_t, \widehat{p}_t)$ and its partial derivatives by G_t, G_X, G_{XX} and G_p once again. Taking the drift of the incomplete market budget restriction from (8), the drift of the filter equation from (1) and the squared volatility of the budget restriction from (8), the Hamilton-Jacobi-Bellman equation reads

$$\max\Big\{G_t + G_X \cdot \mathcal{A}^* X_t + G_p \cdot \mathcal{B} + 0.5 G_{XX} \cdot \mathcal{C}^* X_t^2 \\ + \left[G\left(t, X_A^+(t,\widehat{p}_t), \widehat{p}_A^+\right) - G(t, X_t, \widehat{p}_t)\right]\widehat{\lambda}_A + \left[G\left(t, X_B^+(t,\widehat{p}_t), \widehat{p}_B^+\right) - G(t, X, \widehat{p}_t)\right]\widehat{\lambda}_B\Big\} = 0$$

with

$$\begin{aligned}
\mathcal{A}^* &= r + \pi_A \cdot (\widehat{\mu}_A - r) + \pi_B \cdot (\widehat{\mu}_B - r) \\
\mathcal{B} &= (1-\widehat{p}_t)\lambda^{cont,calm} - \widehat{p}_t(\lambda_A^{calm,cont} + \lambda_B^{calm,cont}) \\
&\quad + \left(\widehat{\lambda}_A - \lambda_A^{calm,calm} + \widehat{\lambda}_B - \lambda_B^{calm,calm}\right) \cdot \widehat{p}_t \\
\mathcal{C}^* &= \pi_A^2 \widehat{\sigma}_A^2 + \pi_B^2 \widehat{\sigma}_B^2 + 2\pi_A \pi_B \rho \widehat{\sigma}_A \widehat{\sigma}_B
\end{aligned}$$

As in the complete market, the second-order partial derivatives with respect to p vanish since the filter equation contains only drift terms and jump processes. The variables X_i^+ and $\widehat{p}_i^+$ again denote the wealth and the subjective probability immediately after a jump in stock i. While, for p^+, we can still use the definition

$$\widehat{p}_i^+ = \widehat{p}_t \frac{\lambda_i^{calm,calm}}{\widehat{p}_t\left(\lambda_i^{calm,calm} + \lambda_i^{calm,cont}\right) + (1-\widehat{p}_t)\lambda_i^{cont,cont}} = \widehat{p}_t \frac{\lambda_i^{calm,calm}}{\widehat{\lambda}_i}$$

from above, the investor's wealth shortly after a jump has to be restated in terms of π_A and π_B:

$$X_i^+ = (1 - \pi_i \cdot L_i) \cdot X_t.$$

Note that the dependence of π_i, $\widehat{\mu}_i$, $\widehat{p}_i^+$ and $\widehat{\lambda}_i$ on $\widehat{p}_t$ (and t) will again be suppressed in the notation for the sake of readability. Plugging in the traditional guess $G(t, X_t, \widehat{p}_t) = \frac{X_t^{1-\gamma}}{1-\gamma} f(t, \widehat{p}_t)$, wealth again factors out and we arrive at

$$\begin{aligned}
\max\{&f_t(t,\widehat{p}_t) + f(t,\widehat{p}_t) \cdot (1-\gamma) \cdot \mathcal{A}^* + f_p(t,\widehat{p}_t) \cdot \mathcal{B} - 0.5\gamma(1-\gamma) \cdot \mathcal{C}^* \cdot f(t,\widehat{p}_t) \\
&+ \left[(1-\pi_A L_A)^{1-\gamma} \cdot f(t,\widehat{p}_A^+) - f(t,\widehat{p}_t)\right]\widehat{\lambda}_A \\
&+ \left[(1-\pi_B L_B)^{1-\gamma} \cdot f(t,\widehat{p}_B^+) - f(t,\widehat{p}_t)\right]\widehat{\lambda}_B\Big\} = 0. \qquad (19)
\end{aligned}$$

Deriving with respect to π_A and π_B gives the first-order conditions

$$\begin{aligned}
0 &= f(t,\widehat{p}_t) \cdot (\widehat{\mu}_A - r) - \gamma \pi_B \rho \widehat{\sigma}_A \widehat{\sigma}_B \cdot f(t,\widehat{p}_t) - \gamma \widehat{\sigma}_A^2 \pi_A \cdot f(t,\widehat{p}_t) \\
&\quad - L_A \cdot (1-\pi_A L_A)^{-\gamma} \cdot f(t,\widehat{p}_A^+) \cdot \widehat{\lambda}_A \qquad (20) \\
0 &= f(t,\widehat{p}_t) \cdot (\widehat{\mu}_B - r) - \gamma \pi_A \rho \widehat{\sigma}_A \widehat{\sigma}_B \cdot f(t,\widehat{p}_t) - \gamma \widehat{\sigma}_B^2 \pi_B \cdot f(t,\widehat{p}_t) \\
&\quad - L_B \cdot (1-\pi_B L_B)^{-\gamma} \cdot f(t,\widehat{p}_B^+) \cdot \widehat{\lambda}_B. \qquad (21)
\end{aligned}$$

Equations (19), (20) and (21) form the system given in proposition 3.2 which completes the proof.

References

Aase, K., 1984, Optimum Portfolio Diversification in a General Continuous-Time Model, *Stochastic Processes and their Applications* 18, 81–98.

Ait-Sahalia, Y., 2004, Disentangling diffusion from jumps, *Journal of Financial Economics* 74, 487–528.

Ait-Sahalia, Y., J. Cacho-Diaz, and T. Hurd, 2009, Portfolio Choice with Jumps: A closed-form Solution, *Annals of Applied Probability* 19, 556–584.

Ait-Sahalia, Y., and J. Jacod, 2009, Testing for jumps in a discretely observed process, *Annals of Statistics* 37, 184–222.

Ang, A., and G. Bekaert, 2002, International Asset Allocation with Regime Shifts, *Review of Financial Studies* 15, 1137–1187.

Branger, N., H. Kraft, and C. Meinerding, 2009, What is the Impact of Stock Market Contagion on an Investor's Portfolio Choice?, *Insurance: Mathematics and Economics* 45, 92–112.

Branger, N., C. Schlag, and E. Schneider, 2008, Optimal Portfolios when Volatility can jump, *Journal of Banking and Finance* 32, 1087–1097.

Brémaud, P., 1981, *Point Processes and Queues: Martingale Dynamics.* Springer, New York.

Broadie, M., M. Chernov, and M. Johannes, 2007, Model Specification and Risk Premia: Evidence from Futures Options, *Journal of Finance* 62, 1453–1490.

Bäuerle, N., and U. Rieder, 2005, Portfolio Optimization with Unobservable Markov-Modulated Drift Process, *Journal of Applied Probability* 42, 362–378.

Bäuerle, N., and U. Rieder, 2007, Portfolio optimization with jumps and unobservable intensity process, *Mathematical Finance* 17, 205–224.

Buraschi, A., P. Porchia, and F. Trojani, 2009, Correlation Risk and Optimal Portfolio Choice, *Working Paper.*

Das, S., and R. Uppal, 2004, Systemic Risk and International Portfolio Choice, *Journal of Finance* 59, 2890–2834.

Delbaen, F., and W. Schachermayer, 1994, A General Version of the Fundamental Theorem of Asset Pricing, *Mathematische Annalen* 300, 463–520.

Detemple, J. B., 1986, Asset pricing in a production economy with incomplete information, *Journal of Finance* 41, 383–391.

Dothan, M.U., and D. Feldman, 1986, Equilibrium interest rates and multiperiod bonds in a partially observable economy, *Journal of Finance* 41, 369–382.

Eraker, B., M. Johannes, and N. Polson, 2003, The Impact of Jumps in Volatility and Returns, *Journal of Finance* 58, 1269–1300.

Framstad, N., B. Oksendal, and A. Sulem, 2001, Optimal Consumption and Portfolio in a Jump Diffusion Market, *Working Paper*.

Guidolin, M., and A. Timmermann, 2005, Optimal Portfolio Choice under Regime Switching, Skew and Kurtosis Preferences, *Working Paper*.

Guidolin, M., and A. Timmermann, 2007, Asset Allocation under Multivariate Regime Switching, *Journal of Economic Dynamics and Control* 31, 3503–3544.

Guidolin, M., and A. Timmermann, 2008, International Asset Allocation under Regime Switching, Skew and Kurtosis Preferences, *Review of Financial Studies* 21, 889–935.

Hamilton, J., 1989, A New Approach to the Economic Analysis of Nonstationary Time Series and the Business Cycle, *Econometrica* 57, 357–384.

Harrison, J., and D.M. Kreps, 1979, Martingales and Arbitrage in Multiperiod Security Markets, *Journal of Economic Theory* 20, 381–408.

Haussmann, U., and J. Sass, 2004, Optimizing the terminal wealth under partial information: The drift process as a continuous time Markov chain, *Finance and Stochastics* 8, 553–577.

Honda, T., 2003, Optimal Portfolio Choice for Unobservable and Regime-Switching Mean Returns, *Journal of Economic Dynamics and Control* 28, 45–78.

Jeanblanc-Picqué, M., and M. Pontier, 1990, Optimal Portfolio for a Small Investor in a Market Model with Discontinuous Prices, *Applied Mathematics and Optimization* 22, 287–310.

Johannes, M., N. Polson, and J. Stroud, 2009, Optimal filtering of jump diffusions: extracting latent states from asset prices, *Review of Financial Studies* 22, 2759–2799.

Kole, E., K. Koedijk, and M. Verbeek, 2006, Portfolio Implications of Systemic Crises, *Journal of Banking and Finance* 30, 2347–2369.

Kraft, H., and M. Steffensen, 2008, How to Invest Optimally in Corporate Bonds: A Reduced-Form Approach, *Journal of Economic Dynamics and Control* 32, 348–385.

Kraft, H., and M. Steffensen, 2009, Asset Allocation with Contagion and Explicit Bankruptcy Procedures, *Journal of Mathematical Economics* 45, 147–167.

Liu, J., F. Longstaff, and J. Pan, 2003, Dynamic Asset Allocation with Event Risk, *Journal of Finance* 58, 231–259.

Liu, J., and J. Pan, 2003, Dynamic Derivative Strategies, *Journal of Financial Economics* 69, 401–430.

Merton, R., 1969, Lifetime Portfolio Selection under Uncertainty: The Continuous-Time Case, *Review of Economics and Statistics* 51, 247–257.

Merton, R., 1971, Optimum Consumption and Portfolio Rules in a Continuous-Time Model, *Journal of Economic Theory* 3, 373–413.

Morton, K.W., and D.F. Mayers, 2005, *Numerical Solution of Partial Differential Equations: An Introduction.* Cambridge University Press, Cambridge.

Prokopczuk, M., 2008, Portfolio Management in the Presence of Systemic Risk, *Working Paper.*

Wu, L., 2003, Jumps and Dynamic Asset Allocation, *Review of Quantitative Finance and Accounting* 20, 207–243.

		Benchmark (equal stocks)	Different stocks	
			Stock A	Stock B
Data-generating process	σ_i^{calm}	0.15	0.15	0.15
	σ_i^{cont}	0.15	0.15	0.15
	ρ^{calm}	0.50	0.50	0.50
	ρ^{cont}	0.50	0.50	0.50
	$\lambda_i^{calm,calm}$	0.32	0.32	0.20
	$\lambda_i^{calm,cont}$	*0.08*	*0.08*	*0.20*
	$\lambda_i^{cont,cont}$	*2.00*	*2.00*	*1.00*
	$\lambda^{cont,calm}$	*1.00*	*0.75*	
	$L_i^{calm,calm}$	0.04	0.04	0.04
	$L_i^{calm,cont}$	0.04	0.04	0.04
	$L_i^{cont,cont}$	0.04	0.04	0.04
	$L_i^{cont,calm}$	0.00	0.00	0.00
	ξ_i	5.00	5.00	2.50
	α_i	0.20	0.20	0.50
	ψ	0.25	0.25	
Market prices of risk	η_i^{calm}	0.35	0.35	0.35
	η_i^{cont}	*0.35*	*0.35*	*0.35*
	$\eta_i^{calm,calm}$	2.00	2.00	2.00
	$\eta_i^{calm,cont}$	*17.0*	*17.0*	*8.00*
	$\eta_i^{cont,cont}$	*0.20*	*0.20*	*1.40*
	$\eta^{cont,calm}$	0.00	0.00	0.00
Risk premia	diffusion risk calm state	*0.0525*	*0.0525*	*0.0525*
	diffusion risk contagion state	*0.0525*	*0.0525*	*0.0525*
	jump risk calm state	0.08	0.08	0.08
	jump risk contagion state	*0.016*	*0.016*	*0.056*

Table 1: Benchmark parameters

The table gives the benchmark parametrization of our model. The first column shows the benchmark parameters in the case of equal stocks (section 4.2). The second and third column give the parameters for the case where both stocks differ with respect to ξ and α (section 4.3). The italic numbers denote those parameters which are not chosen freely, but follow from the other ones.

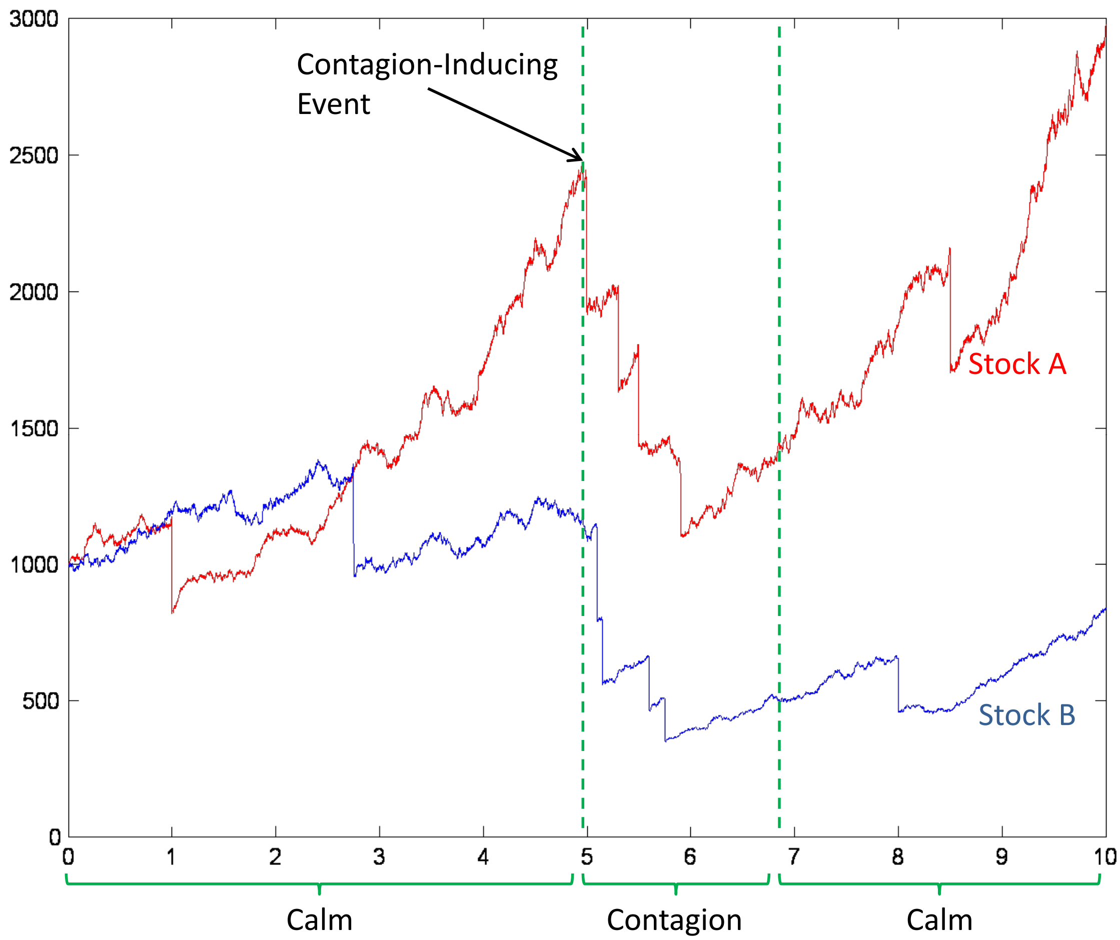

Figure 1: Typical sample path

The figure shows a typical sample path in our model. Both stocks (A and B) follow jump-diffusion processes and are subject to the risk of downward jumps. In the example, the downward jump in stock A at time $t = 5$ triggers contagion. Jump probabilities for both stocks are significantly larger until the economy leaves the contagion state again at time $t = 7$. While there is a loss in one stock when the economy enters the contagion state, jumps back to the calm state have no direct impact on the stock prices.

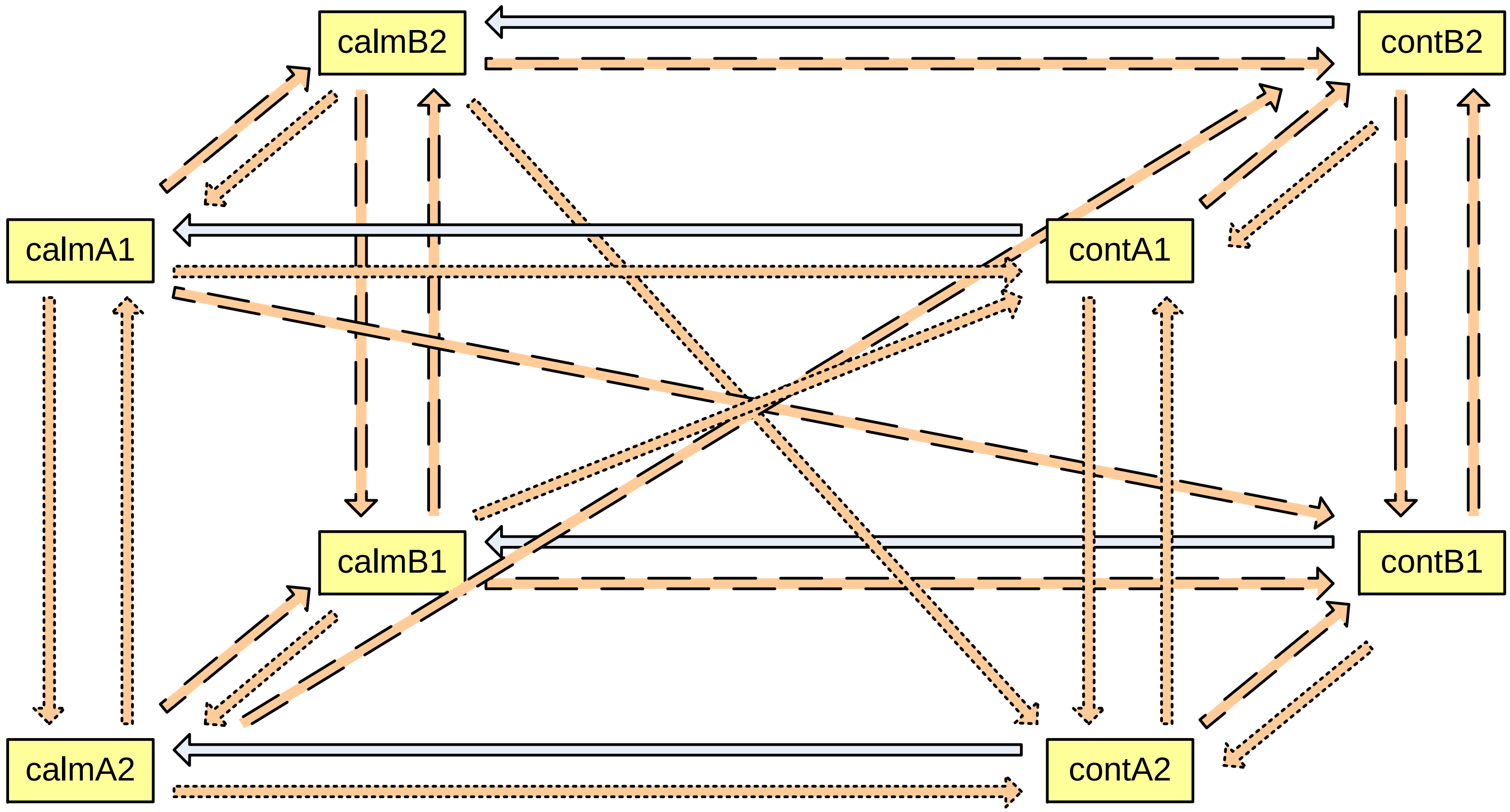

Figure 2: Hidden Markov chain

This figure shows the hidden Markov chain which drives the economy. The states on the left hand side denote the calm states of the economy, while the ones on the right hand side denote those states in which the stocks are affected by contagion. The dotted (orange) arrows indicate a jump event leading to a loss in stock A, the dashed (orange) arrows a jump event leading to a loss in stock B. The solid (blue) arrows denote a change of state without any impact on the stock prices, i.e. a jump from the contagion state back to the calm state. Missing arrows between two states indicate that the corresponding jump intensities are zero.

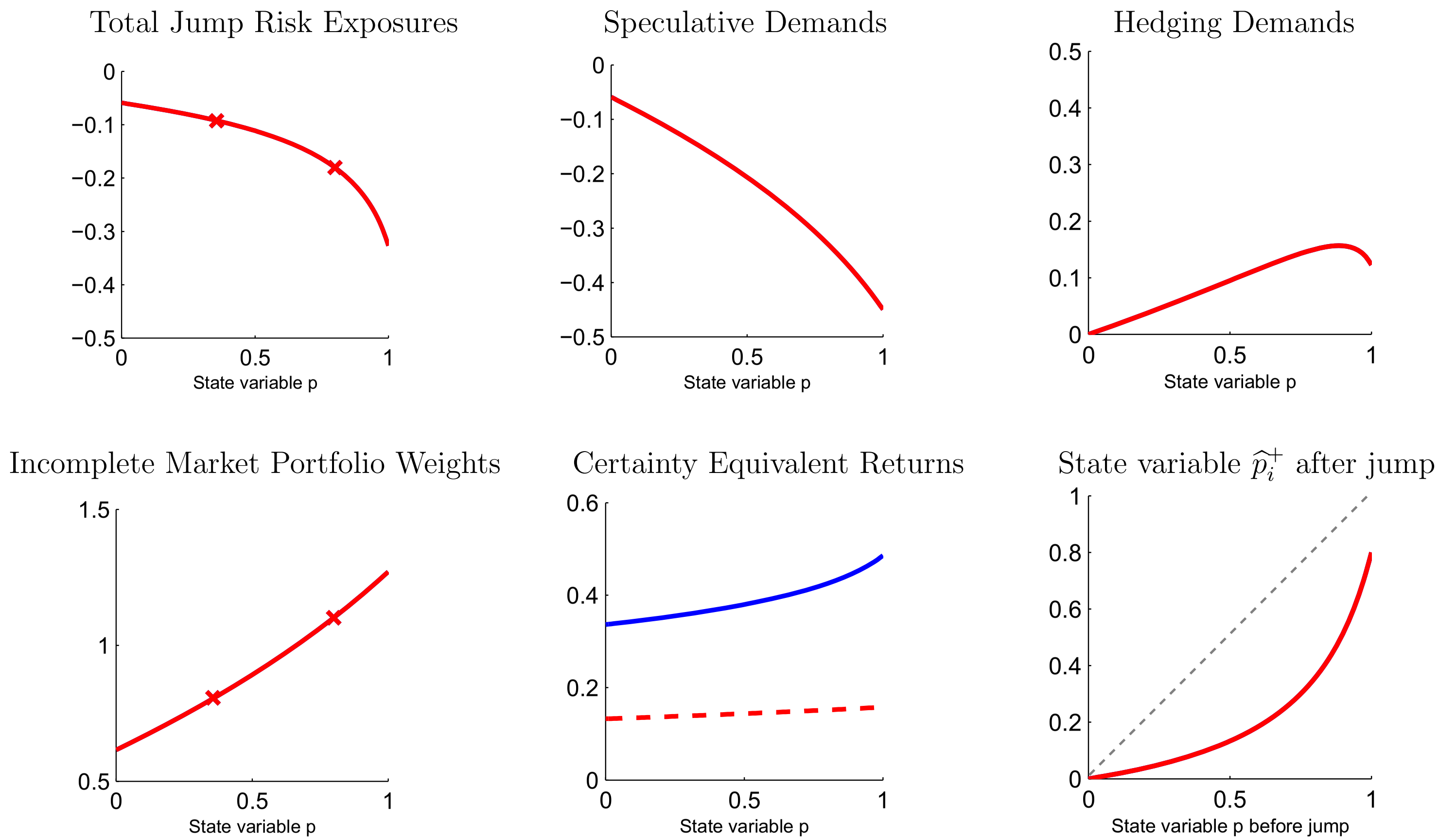

Figure 3: Solution to the portfolio planning problem with identical stocks

The figure gives the solution in the benchmark case with identical stocks, for which the parameters can be found in the first column of Table 1. The upper left graph gives the optimal demand for jump risk in a complete market as a function of the state variable $\widehat{p}_t$. The upper middle and right graph show the speculative demand and the hedging demand, respectively. The lower left graph gives the optimal portfolio weights for the stocks in the incomplete market as a function of $\widehat{p}_t$. The lower middle graph shows the certainty equivalent returns in the complete market (solid blue line) and in the incomplete market (dashed red line). The lower right graph shows the updated state variable $\widehat{p}_i^+$ after a jump as a function of $\widehat{p}_t$ before the jump. The crosses in the graphs in the left row refer to this graph: they mark the subjective probability $\widehat{p}_i^+$ after one and two jumps if $\widehat{p}_t$ was equal to 1 initially.

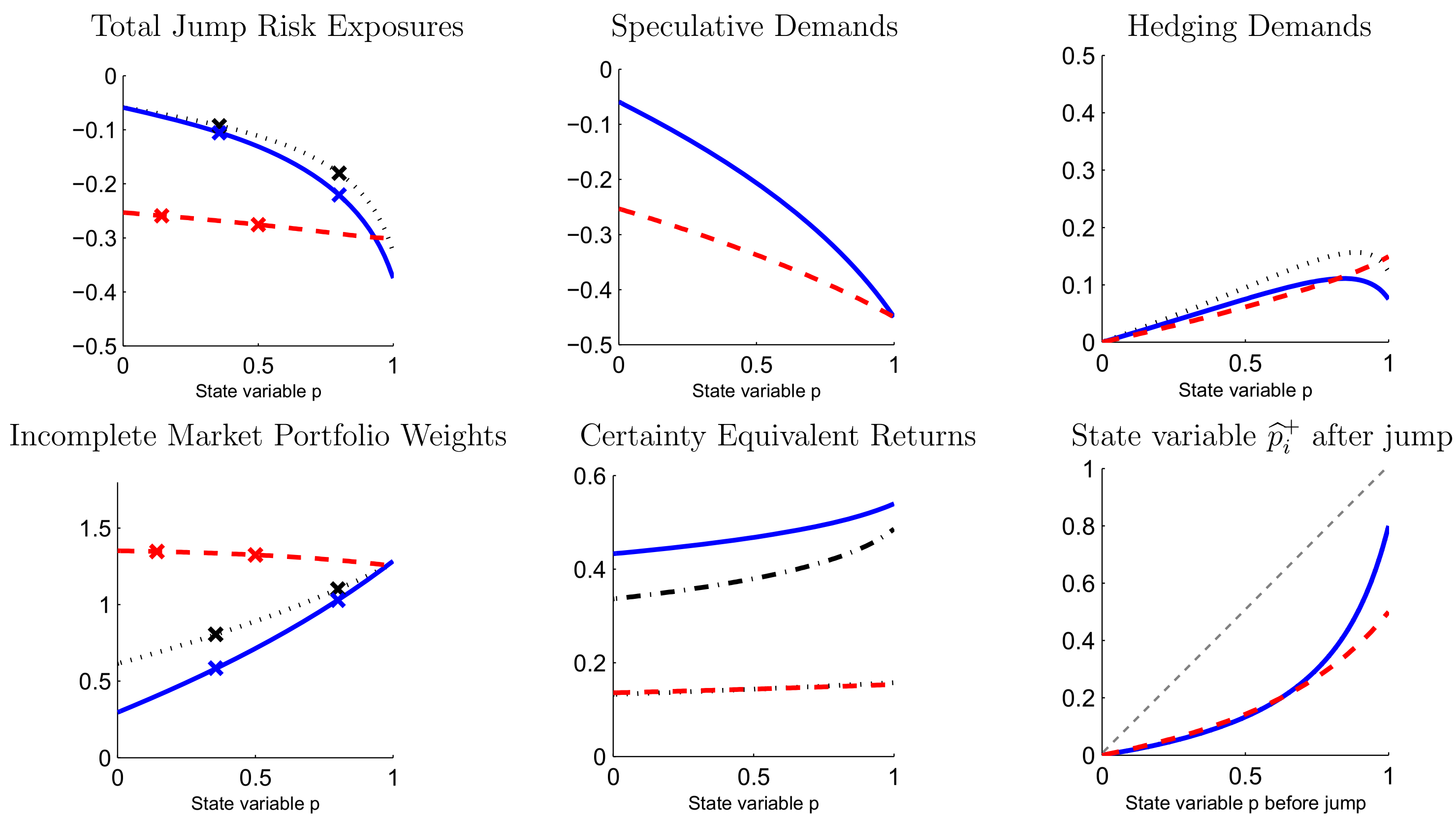

Figure 4: Solution to the portfolio planning problem with different stocks

The figure gives the solution when the stocks differ with respect to the jump parameters, for which the parameters can be found in the second and third column of Table 1. The ordering of the graphs is as in Figure 3. In the four graphs which show the demand, we give the solution for stock A (solid blue line) and stock B (dashed red line) as well as the solutions in the benchmark case (dotted black line). In the middle lower graph which shows the CER, we give the solution in the complete market (solid blue line) and in the incomplete market (dashed red line) as well as the benchmark solutions in the complete market (dash-dotted black line) and in the incomplete market (dotted black line). In the left lower graph, we give the updated probabilities after a jump in stock A (solid blue line) and stock B (dashed red line). The crosses in the graphs in the left row refer to this graph: they mark the subjective probabilities $\widehat{p}_i^+$ after one and two jumps if $\widehat{p}_t$ was equal to 1 initially. Note that we distinguish between the effect of jumps in stock A (blue crosses), jumps in stock B (red crosses), and jumps in the benchmark case (black crosses).

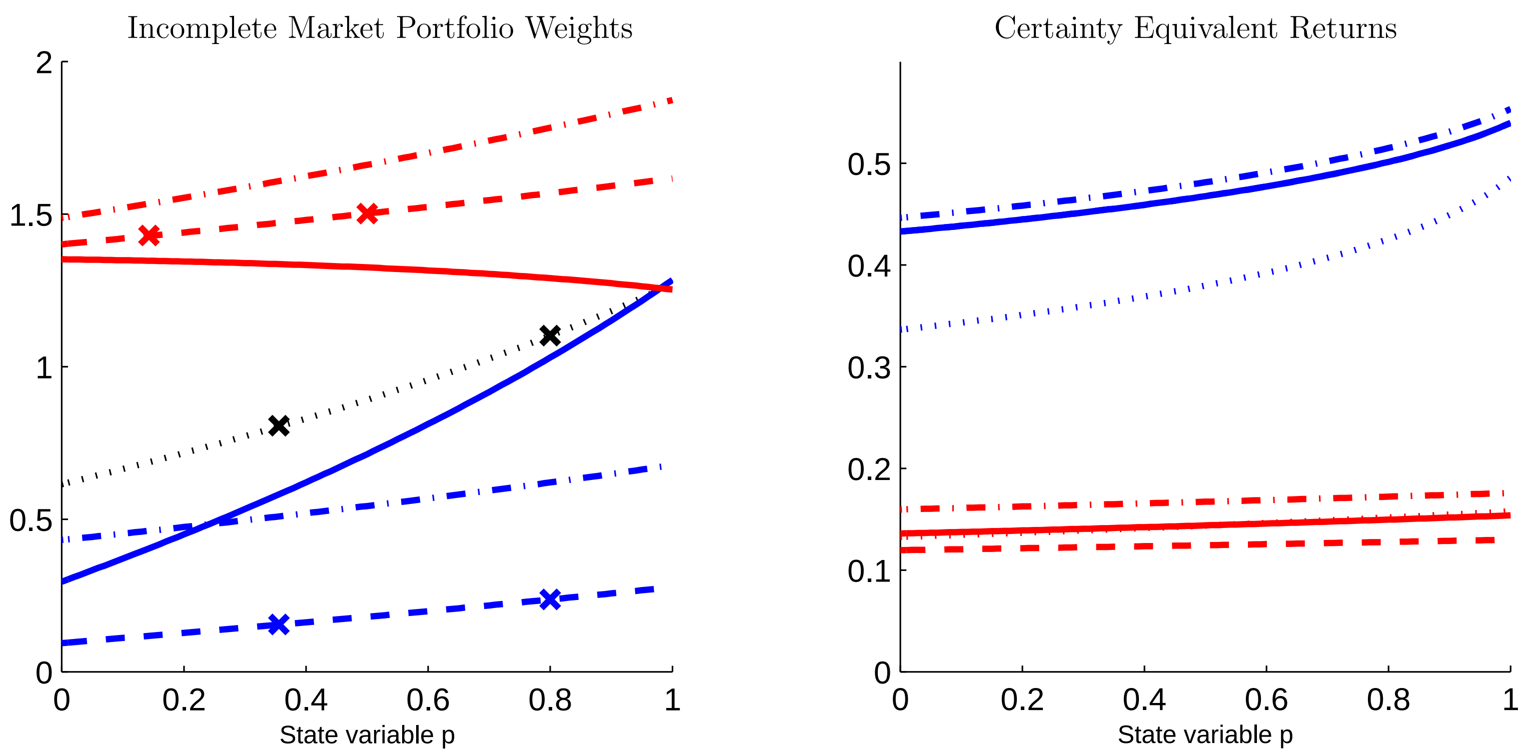

Figure 5: Diffusion versus jump risk

The figure gives the optimal portfolio weights in the incomplete market (left graph) and the CER (right graph) for the parameterizations described in Section 4.4. The left graph gives the optimal demand in the benchmark case with equal stocks (black dotted line), in the case with different stocks (solid lines) described in Section 4.3, in the case with a larger volatility of stock A (dashed lines), and in the case where additionally $\rho = 0$ (dash-dotted lines). The lower blue lines give the demand for stock A, the upper red lines give the demand for stock B. The right graph gives the CER in the complete market (blue lines) and in the incomplete market (red lines). The line forms (straight, dashed, etc.) have the same meaning as in the left graph.