Modeling of Contagious Credit Events and Risk Analysis of Collateralized Debt Obligations

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Abstract

We present a new model of credit events such as rating changes as well as defaults for risk analyses of some portfolio credit derivatives. The framework of our model is based on a so-called top-down approach. To be precise, we firstly pay attention to modeling the point process of each type of credit events in the whole economy with a self-exciting intensity process, and then we characterize the point processes of credit events for the underlying sub-portfolio with some random thinning processes specified by the distribution of credit ratings in the sub-portfolios. One of the main features of our model is that the model can capture credit risk contagion among several credit portfolios. We show a credit event simulation algorithm based on our model and illustrate as an application of the model some numerical examples of risk analysis of collateralized debt obligations (CDO) and CDO-squared.

Keywords: Credit risk, rating change, top-down approach, self-exciting intensity model, state-dependent, CDO, CDO-squared

1 Introduction

This paper presents a new model of the intensities of contagious credit events such as rating changes and defaults. Our modeling framework is based on the top-down approach studied in Giesecke and Goldberg [3] and Nakagawa [5, 6]. That is, we apply self-exciting stochastic processes to model the intensities of credit events in the economy. Then, we apply a random thinning to specify the intensities of the events in a sub-portfolio. Our model enables us to measure risks of several portfolios simultaneously with credit risk contagion.

Giesecke and Goldberg [3] introduced the top-down approach to evaluate the default risk of the portfolio. In the top-down approach, intensity-based models of the loss point process are specified without reference to the portfolio constituents. In addition, the method called random thinning decomposes the portfolio loss point process into the sum of its constituent loss point processes. In particular, Giesecke and Goldberg [3] proposed the portfolio default intensity model with self-exciting effect. The word “self-exciting” means that the intensity of next default jumps due to happening of default. A self-exciting default intensity model is one of the models that reflect default contagion.

There are several works on risk analysis and pricing of portfolio credit derivatives within the framework of Giesecke and Goldberg [3]. For instance, Giesecke and Kim [4] analyzed risks of collateralized debt obligations (CDOs) within the framework. They used self-exciting intensity process to model defaults of the underlying credit portfolio of a CDO. In particular, their intensity process has not only self-exciting effect but also state-dependent property. In addition, Giesecke and Kim [4] developed a data generating method to get the default timing data of reference credit portfolio. The data generating method, called acceptance/rejection re-sampling scheme, is based on the random thinning.

Nakagawa [5, 6] proposed a rating change intensity model within the top-down approach, while Giesecke and Goldberg [3] and Giesecke and Kim [4] focused on default intensity model. Nakagawa [5] proposed the self-exciting intensity model for both defaults and rating changes such as up-grades and down-grades. Nakagawa [6] proposed the intensity model that has not only self-exciting property but also mutually exciting property. Moreover, Nakagawa [5, 6] proposed the credit derivative named multi-downgrade protection and mentioned their model is efficient for pricing of the multi-downgrade protection. As Nakagawa [6] noted, rating changes are usually modeled by rating transition intensity matrix, but it is difficult to use the rating transition matrix framework so as to consider dynamic risk dependence in the portfolio. Although, with the model of Nakagawa [5, 6], we can consider dynamic risk dependence in the portfolio without difficulty.

This paper provides a new model of credit events in the whole economy and thinning to measure risk of portfolio credit derivatives. Here, credit events are rating changes (i.e. up-grades and down-grades) and defaults. Our model is an extension of the model proposed by Nakagawa [5]. In addition, we propose an event simulation algorithm based on our model. The algorithm can work for other models as well as our intensity model.

We model the credit events in the whole economy with state-dependent type self-exciting intensity processes. Due to modeling economy-wide event with some self-exciting intensity, our model can capture credit risk dependence among several credit portfolios. That is, an event occurrence in one portfolio may influence the probability of the next event in the whole economy, and thus may have some impact on the probability of the next event in other portfolios. Our credit event intensity model is analogous to the default intensity model proposed by Giesecke and Kim [4]. However, it should be noted that our intensity model is different in the respect of introducing jump effect from the default intensity model of Giesecke and Kim [4].

As we consider not only defaults but also rating changes, we can capture changes of credit quality of credit portfolios. We specify the thinning with the distribution of credit quality of portfolios, namely with the relative frequency of credit ratings in the bond portfolio. Our model is useful for risk analysis of portfolio credit derivatives such as CDOs. Furthermore, as our model treats several portfolios simultaneously, our model enables to analyze several CDOs simultaneously. In addition, our model is adequate to analyze CDO-squareds.

This paper is organized as follows. Section 2 formulates the point process model. Section 3 develops an event simulation algorithm. In section 4, we give some numerical examples on the risk analysis of cash CDOs and CDO-squareds, and try tentative estimation with historical records on credit events of rated corporate issuers in Japan. Section 5 gives some concluding remarks.

2 Models

In this section, we present our point process model of each type of credit events int the whole economy. In addition, we specify random thinning by the distribution of credit ratings in the sub-portfolios.

2.1 Intensity models for economy-wide events

We will model contagious rating changes and defaults by the point processes with self-exciting intensity processes.

Suppose each firm in the economy is associated with a credit rating. There are $K+1$ ratings and we denote ratings with numbers of $1, 2, \cdots, K$ and $K+1$. The order of ratings represents credit quality, that is, the rating $\rho = 1$ represents the best possible credit quality like AAA, $\rho = K$ represents the worst non-defaulted credit quality and $\rho = K+1$ represents the state of default. Let S^* denote the set of all rated firms, namely the whole economy. Each rated firm belongs to one of sub-portfolios $S_i \ (i = 1, 2, \cdots, I)$. Here we assume $S^* = \bigcup_{i=1}^{I} S_i, \ S_i \cap S_j = \emptyset \ (i \neq j)$.

The uncertainty in the economy is modeled by a filtered complete probability space $(\Omega, \ \mathcal{F}, \ \mathbb{P}, \ \{\mathcal{F}_t\})$, where $\mathbb{P}$ is the actual probability measure. Here $\{\mathcal{F}_t\}$ is a right-continuous and complete filtration. Let $l \in \{1, \ 2, \ \cdots, \ L\}$ denote the types of the credit events. For each l, let $0 < T_1^l < T_2^l < \cdots$ be $\{\mathcal{F}_t\}$-adopted point processes, that is , increasing sequences of $\{\mathcal{F}_t\}$-stopping times.Here we assume $T_n^l < \infty \ a.s. \ {}^\forall n \in \mathbb{N}$. $\{T_n^l\}_{n\in\mathbb{N}}$ indicates the event times of event type l. In addition, we denote the counting process of event l by $N_t^l = \sum_{n\geq 1} \mathbf{1}_{\{T_n^l \leq t\}}$. Furthermore, we assume different types of events do not occur at the same time.

To simplify matters, we focus on rating changes and defaults for the credit events. That is, we set $L = 3$ and suppose the event $l = 1$ indicates up-grade, the event $l = 2$ indicates down-grade and the event $l = 3$ indicates default. Furthermore, we assume each defaulted firm vanishes when the default occurs, and the defaulter is not replaced by a new firm.

Suppose each N_t^l has a strictly positive intensity h_t^l such that $\int_0^t h_s^l \mathrm{d}s < \infty$ almost surely. Namely, each h_t^l is a $\{\mathcal{F}_t\}$-progressively measurable strictly positive process, and the process

$$N_t^l - \int_0^t h_s^l \mathrm{d}s \tag{1}$$

is a $\{\mathcal{F}_t\}$-local martingale. To specify h_t^l, let λ_t^l be the self-exciting stochastic process:

$$\begin{aligned}
\mathrm{d}\lambda_t^l &= \kappa_t^l (c_t^l - \lambda_t^l)\mathrm{d}t + \mathrm{d}J_t^l, &(2)\\
J_t^l &= \sum_{n\geq 1} (\min(\delta^l \lambda_{T_n^l -}^l,\ \gamma^l) \mathbf{1}_{\{T_n^l \leq t\}}), &(3)\\
\kappa_t^l &= \kappa^l \lambda_{T_{N_t^l}^l}^l,\ c_t^l = c^l \lambda_{T_{N_t^l}^l}^l. &(4)
\end{aligned}$$

The quantities $\kappa^l > 0,\ c^l \in (0,1),\ \delta^l > 0,\ \gamma^l \geq 0,\ \lambda_0^l > 0$ are parameters.

With the self-exciting process λ_t^l, we specify intensities of N_t^l , denoted by h_t^l, as follows:

$$\begin{aligned}
h_t^1 &:= \lambda_t^1 \mathbf{1}_{\{\sum_{\rho=2}^{K} X_t^*(\rho) > 0\}}, &(5)\\
h_t^2 &:= \lambda_t^2 \mathbf{1}_{\{\sum_{\rho=1}^{K-1} X_t^*(\rho) > 0\}}, &(6)\\
h_t^3 &:= \lambda_t^3 (\sum_{\rho=1}^{K} z_\rho \mathbf{1}_{\{X_t^*(\rho) > 0\}}). &(7)
\end{aligned}$$

Here, $X_t^*(\rho)$ indicates the number at time t of ρ-rated firms. z_ρ are constants satisfying $0 \leq z_\rho \leq 1$ $(\rho = 1,\ 2, \cdots,\ K)$ and $\sum_{\rho=1}^{K} z_\rho = 1$. Each z_ρ represents the conditional probability that the rating of a defaulter is ρ, given that a default occurs. The indicator function in (5) implies that there are no up-grades when the all firms are rated at the highest rating 1. Also the indicator functions in (6) and (7) imply that there are no down-grades when all firms are rated at the lowest rating K and default intensity reduces by z_ρ if there are no ρ-rated firms in the whole economy. In the rest of this paper, we suppose there are enough number of rated firms in each rating category to regard h_t^l are the same as λ_t^l.

Figure 1 shows a sample pass of state-dependent self-exciting intensity process (2), (3), (4). The intensity processes (2), (3), (4) are analogous to the default intensity model in Giesecke and Kim [4]. Giesecke and Kim [4] introduced the state-dependent self-exciting default intensity process (2), (4) with the jump type of

$$J_t = \sum_{n\geq 1} (\max(\delta \lambda_{T_n -},\ \gamma) \mathbf{1}_{\{T_n \leq t\}}). \tag{8}$$

The remarkable characteristics of both intensity processes (2), (3), (4) and that of Giesecke and Kim [4] are the following. First, the intensity jumps at event times. The jumps of the intensity process generate event correlations. Namely, an event occurrence increases the probability of the next event. This feature facilitates the replication of event clusters. We will mention the details of the jumps later. Second, each event intensity moves deterministically between event times; the intensity jumps at event times. The state-dependent drift involves a reversion level and speed, which are proportional to the intensity at the previous event. Definitely, for $T_n^l \leq t < T_{n+1}^l$, the behavior of the process is described as

$$\lambda_t^l = c^l \lambda_{T_n^l}^l + (1 - c^l)\lambda_{T_n^l}^l \exp(-\kappa^l \lambda_{T_n^l}^l (t - T_n^l)) \quad T_n^l \leq t < T_{n+1}^l. \tag{9}$$

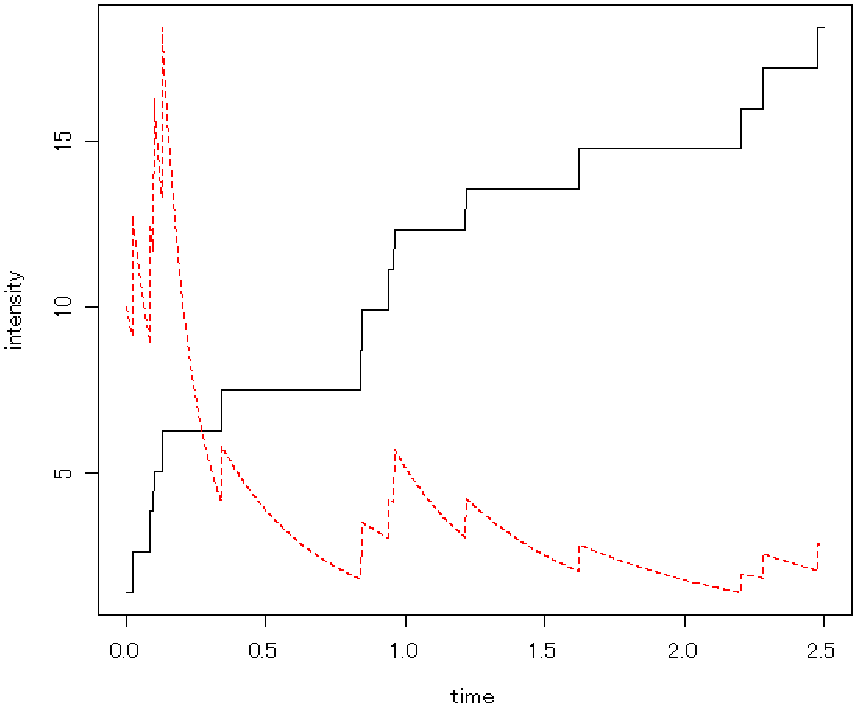

Fig. 1 A sample pass of the self-exciting process (2, 3, 4) and cumulative number of the events (the solid line indicates cumulative number of events, the dash line indicates a pass of self-exciting process (2, 3, 4) with $\kappa = 0.5$, $c = 0.1$, $\delta = 0.4$, $\gamma = 50.0$, $\lambda_0 = 10.0$.)

Now, we should note the difference between our intensity process and the default intensity process proposed by Giesecke and Kim [4]. The difference is that the jump size of our intensity model is bounded, though the jump size of the intensity of Giesecke and Kim [4] is not. We found the estimated intensity model of rating changes with jump part (8) tends to generate unrealistic number of rating changes through simulations. Thus, we propose the intensity model with the limited jump size such as (3).

Figure 2 shows fitted path of the up-grade intensity to the actual data of up-grades. In figure 2, the bar chart is the number of up-grades in 2004, in Japan, and the broken line graph is the estimated paths of the up-grade intensity. Figure 2 shows that the fitted pass of the intensity overlaps considerably with the number of the events. This implies the validity of modeling event clusters with the self-exciting process [*1].

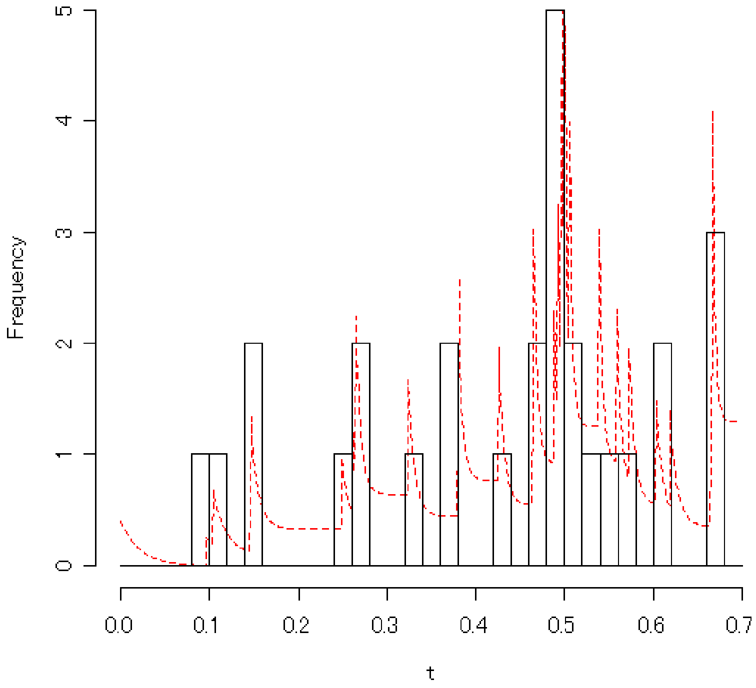

Fig. 2 The number of up-grades and estimated up-grade intensity pass (the ratings are of R&I, parameters were estimated with the maximum likelihood method.)

2.2 Thinning process

We decompose the event intensity for the whole economy into intensities for the underlying sub-portfolio with the random thinning based on rating distributions.

We introduce $\{\mathcal{F}_t\}$-adopted processes $\{Z_t^{(i,1)}(\rho,\ m)\}$, $\{Z_t^{(i,2)}(\rho,\ m)\}$ and $\{Z_t^{(i,3)}(\rho)\}$. $Z_t^{(i,1)}(\rho,\ m)$ represents the conditional probability that the up-grade is the transition from the rating ρ to the rating $\rho - m$ of the firm in the portfolio S_i,

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[*1] Nakagawa [7] find self-exciting effects in rating changes of Japanese firms during April 1998 to April 2009 with mutually exciting intensity model, which is a extension of the self-exciting models.

given that an up-grade occurs in the economy. Similarly, $Z_t^{(i,2)}(\rho,\ m)$ represents the conditional probability that the down-grade is the transition from the rating ρ to the $\rho+m$ of the firm in the portfolio S_i, given that the down-grade occurs in the economy. Also, $Z_t^{(i,3)}(\rho)$ represents the conditional probability that the defaulter is the ρ-rated firm in the portfolio S_i, given that the default occurs in the economy.

From proposition in Giesecke and Goldberg [3], $Z_t^{(i,l)}(\rho,\ m)$ $(l=1,2)$, are described definitely as

$$\begin{aligned}
&Z_t^{(i,l)}(\rho,m)=\lim_{\epsilon\to 0} Z_t^{(i,l)}(\rho,m,\epsilon),\\
&Z_t^{(i,l)}(\rho,m,\epsilon)\\
&\quad=\sum_n \frac{P[\{T_n^l\in\tau^l(S_i)\}\cap\{T_n^l\in\tau^l(\rho)\}\cap\{T_n^l\in\tau^l(m)\}\cap\{T_n^l\le t+\epsilon\}\mid\mathcal{F}_t]}{P[T_n^l\le t+\epsilon\mid\mathcal{F}_t]}\mathbf{1}_{\{T_{n-1}^l<t\le T_n^l\}},
\end{aligned}$$

and $Z_t^{(i,3)}(\rho)$ is described definitely as

$$\begin{aligned}
Z_t^{(i,3)}(\rho)&=\lim_{\epsilon\to 0} Z_t^{(i,3)}(\rho,\epsilon),\\
Z_t^{(i,3)}(\rho,\epsilon)&=\sum_n \frac{P[\{T_n^3\in\tau^3(S_i)\}\cap\{T_n^3\in\tau^3(\rho)\}\cap\{T_n^3\le t+\epsilon\}\mid\mathcal{F}_t]}{P[T_n^3\le t+\epsilon\mid\mathcal{F}_t]}\mathbf{1}_{\{T_{n-1}^3<t\le T_n^3\}}.
\end{aligned}$$

Here $\tau^l(S_i)$ denote the set of event times of event l in the portfolio S_i, $\tau^l(m)(l=1,2)$ denote the set of rating change times of m-step rating change, and $\tau^l(\rho)$ denote the event times of event l of ρ-rated firms. Also, $\epsilon>0$ and the quotients on the right are taken to be 0 when the denominator vanishes. In addition, $Z_t^{(i,l)}(\rho)$ satisfy the following property: for each l, (1) $Z_t^{(i,l)}(\rho)$ takes values in the unit interval $[0,1]$,(2) $\sum_{i,\rho} Z_t^{(i,l)}(\rho)=1$.

Let $N_t^{(i,1)}(\rho,\ m)$ be the counting process of the m-step up-grades of ρ-rated firms in portfolio S_i. Also, let $N_t^{(i,2)}(\rho,\ m)$ be the counting process of the m-step down-grades of ρ-rated firms in portfolio S_i and $N_t^{(i,3)}(\rho)$ be the counting process of the defaults of ρ-rated firms in portfolio S_i. These counting processes are given by $N_t^{(i,l)}(\rho,\ m)=\sum_{n\ge 1}\mathbf{1}_{\{T_n^l\le t\}\cap\{T_n^l\in\tau^l(S_i)\}\cap\{T_n^l\in\tau^l(m)\}\cap\{T_n^l\in\tau^l(\rho)\}}$ $(l=1,2)$ and $N_t^{(i,3)}(\rho,\ m)=\sum_{n\ge 1}\mathbf{1}_{\{T_n^3\le t\}\cap\{T_n^3\in\tau^3(S_i)\}\cap\{T_n^3\in\tau^3(\rho)\}}$. With $Z_t^{(i,1)}(\rho,\ m), Z_t^{(i,2)}(\rho,\ m)$ and $Z_t^{(i,3)}(\rho)$, we obtain the intensities of $N_t^{(i,1)}(\rho,\ m)$, $N_t^{(i,2)}(\rho,\ m)$ and $N_t^{(i,3)}(\rho)$ as following: $\lambda_t^{(i,1)}(\rho,\ m)=Z_t^{(i,1)}(\rho,\ m)\lambda_t^1$, $\lambda_t^{(i,2)}(\rho,\ m)=Z_t^{(i,2)}(\rho,\ m)\lambda_t^2$ and $\lambda_t^{(i,3)}(\rho)=Z_t^{(i,3)}(\rho)\lambda_t^3$.

We assume that $Z_t^{(i,l)}(\rho,\ m)$ $(l=1,2)$ and $Z_t^{(i,3)}(\rho)$ are determined by the distribution of ratings in the portfolios. That is, we specify $Z_t^{(i,l)}(\rho,\ m)(l=1,2)$ and

$Z_t^{(i,3)}(\rho)$ as following:

$$Z_t^{(i,1)}(\rho,\ m) = \frac{X_t^{(i)}(\rho)}{\sum_{\rho=1+m}^{K} X_t^*(\rho)} z_m^1 \mathbf{1}_{\{\sum_{\rho=1+m}^{K} X_t^*(\rho)>0\}} \quad (\rho = 1+m, 2+m, \cdots, K), \tag{10}$$

$$Z_t^{(i,2)}(\rho,\ m) = \frac{X_t^{(i)}(\rho)}{\sum_{\rho=1}^{K-m} X_t^*(\rho)} z_m^2 \mathbf{1}_{\{\sum_{\rho=1}^{K-m} X_t^*(\rho)>0\}} \quad (\rho = 1, 2, \cdots, K-m), \tag{11}$$

$$Z_t^{(i,3)}(\rho) = \frac{X_t^{(i)}(\rho)}{X_t^*(\rho)} z_\rho \mathbf{1}_{\{X_t^*(\rho)>0\}} \quad (\rho = 1, 2, \cdots, K). \tag{12}$$

Here, $X_t^{(i)}(\rho)$ denotes the number at time t of ρ-rated firms in the portfolio S_i. As we have mentioned earlier, $X_t^*(\rho)$ denotes the number at time t of ρ-rated firms in the whole economy. $z_m^l(l = 1, 2,\ m = 1,\ 2,\ \cdots,\ K-1)$ denote conditional probability that the rating change is the m step rating change, given that the rating change occurs. $z_m^l(i = 1, 2,\ m = 1,\ 2,\ \cdots,\ K-1)$ are constants and satisfies $0 \leq z_m^l \leq 1$ $(l = 1,\ 2)$ and $\sum_{m=1} z_m^l = 1$. Recall that each z_ρ represents the conditional probability that the rating of a defaulter is ρ, given that a default occurs and z_ρ are constants satisfying $0 \leq z_\rho \leq 1$ $(\rho = 1,\ 2, \cdots,\ K), \sum_{\rho=1}^{K} z_\rho = 1$.

3 Event time Simulation

This section shows an event simulation algorithm based on the model introduced in the section 2. We should note that our algorithm can work for other models as well as our intensity model.

In our algorithm, we generate candidate event times in the step 2 and decide to accept or to reject the candidate event times in the step 3. The step 2 is based on the point process simulation method proposed by Ogata [8]. In the step 3, we determine details of the event time along with the thinning processes.

Algorithm: Generating event times over $[0, H]$ for a horizon H and the model in the section 2

1. **[Set values of the parameters and initialize simulation settings]**
    - Set $(\kappa^l,\ c^l,\ \delta^l,\ \gamma^l,\ \lambda_0^l)$ $(l = 1,\ 2,\ 3)$ and z_ρ $(\rho = 1,\ 2,\ \cdots,\ K)$.
    - Initialize $S = 0$, which indicate the present time. Set the time horizon $H (> 0)$.
    - Set $X_T^{(i)}(\rho) = X_0^{(i)}(\rho)$ $(i = 1,\ 2,\ \cdots,\ I, \quad \rho = 1,\ 2,\ \cdots,\ K)$, the number of ρ-rated firms in the portfolio S_i.
    - Initialize the cumulative event number $N^l = 0$ $(l = 1,\ 2,\ 3)$.
    - Initialize the last event time $T_{N^l}^l = 0$ $(l = 1,\ 2,\ 3)$.

2. **[Generate a candidate event time T]**
    - Set $\Lambda = \sum_{l=1}^{3} \lambda_S^l$. Draw $\mathcal{E} \sim \exp(\Lambda)$.

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  • Set $T = S + \mathcal{E}$. If $T > H$, stop.

3. [Decide to accept or to reject the candidate event time $T$]
  • For $l = 1,\ 2,\ 3$, evaluate $\lambda_T^l = c^l \lambda^l_{T^l_{N^l}} + (\lambda^l_S - c^l \lambda^j_{T^l_{N^l}}) \exp(-\kappa^l \lambda^l_{T^l_{N^l}} (T - S))$.
  • Draw $u_1 \sim U(0,1)$.
  • If $u_1 < \lambda_T^1/\Lambda$ and $\sum_{\rho=2}^{K} \sum_{i=1}^{I} X_T^{(i)}(\rho) > 0$,
        then set $l = 1$. (Accept $T$ as a event time of the event type 1)
    Else if $u_1 < (\lambda_T^1 + \lambda_T^2)/\Lambda$ and $\sum_{\rho=1}^{K-1} \sum_{i=1}^{I} X_T^{(i)}(\rho) > 0$,
        then set $l = 2$. (Accept $T$ as a event time of the event type 2)
    Else if $u_1 < (\lambda_T^1 + \lambda_T^2 + \lambda_T^3)/\Lambda$ and $\sum_{\rho=1}^{K} \sum_{i=1}^{I} X_T^{(i)}(\rho) > 0$,
        then set $l = 3$. (Accept $T$ as a event time of the event type 3)
    Else, go to 6.    (Reject $T$)

4. [Updating]
  For the event $l$ accepted in the step 3,
  • $N^l = N^l + 1$, (Update cumulative number of event $l$)
  • $T^l_{N^l} = T$, (Update the last event $l$ time)
  • $\lambda_T^l = \lambda_T^l + \delta^l \lambda_T^l$. (Update the intensity $\lambda^l$)

5. [Thinning]
  • Draw $u_2 \sim U(0,1)$.
  • Evaluate $Z_T^{(i,l)}(\rho, m),\ Z_T^{(i,3)}(\rho)(i = 1,\ 2,\ \cdots,\ I, \quad \rho = 1,\ 2,\ \cdots,\ K)$ with
    employing (10),(11),(12).
  • Set $Z = 0$.
      If($l = 1, 2$){
        For($m = 1$ to $K - 1$){
          For($i = 1$ to $I$){
            For($\rho = 1$ to $K$){
                $Z = Z + Z_T^{(i,l)}(\rho, m)$.
                If $u_2 < Z$, then
                    · If $l = 1$, then $X_T^{(i)}(\rho) = X_T^{(i)}(\rho) - 1$
                              and $X_T^{(i)}(\rho - m) = X_T^{(i)}(\rho - m) + 1$.
                    · If $l = 2$, then $X_T^{(i)}(\rho) = X_T^{(i)}(\rho) - 1$
                              and $X_T^{(i)}(\rho + m) = X_T^{(i)}(\rho + m) + 1$.
                    · Go to 6.
            }
          }
        }
      }
      Else If ($l = 3$){
        For ($i = 1$ to $I$){
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            For ($\rho = 1$ to $K$){
                $Z = Z + Z_T^{(i,l)}(\rho)$.
                If $u_2 < Z$, then
                    · $X_T^{(i)}(\rho) = X_T^{(i)}(\rho) - 1$ .
                    · Go to 6.
            }
        }
    }
```

6. Set $S = T$ and go to step 2.

4 Numerical Example: Risk Analysis of Collateralized Debt Obligations

In this section, we show some numerical examples on risk analysis of CDOs and a CDO-squared. In addition, we try to estimate the model parameters from the actual data on rating changes in Japan.

4.1 Estimation

In this subsection, we mention about parameter estimation methods for intensity processes and thinning processes respectively. In addition, we present some tentative estimation results with actual data of rating transitions in Japan.

4.1.1 Maximum likelihood method for intensity processes

We estimate the parameter of the intensity model $(\kappa^l, c^l, \delta^l, \gamma^l, \lambda_0^l)$ from the data that consists of the sequence of event data. Let us recall that we suppose that there are enough numbers of firms to identify the self-exciting process λ_t^l with the intensity of the corresponding type of events, namely $h_t^l = \lambda_t^l$. Note that the data we used for the estimation satisfies the assumption.

For the purpose, we apply the maximum likelihood method executed in Giesecke and Kim [4].Suppose that we have event times data of event type l, $0 < T_1^l < T_2^l < \ldots < T_N^l (< H)$. Then the likelihood function of the intensity of the event l is following:

$$\sum_{n=1}^{N} \log \lambda_{T_n^l -}^l - \int_0^H \lambda_s^l \mathrm{d}s. \tag{13}$$

We specify the parameters that maximize (13).

For testing validity of estimated model to the data, we apply the Kolmogorov-Smirnov test, that Azizpour and Giesecke [1] and Nakagawa [7] executed as following.

First, we transform the event times $\{T_n^l\}_{n=1}^N$ into A_n^l by

$$A_n^l := \int_0^{T_n^l} \lambda_s^l \mathrm{d}s. \tag{14}$$

We execute the Kolmogorov-Smirnov test using the fact that $\{A_n^l\}_{n=1}^N$ will be jump times of the standard Poisson process in the case of $\{T_n^l\}_{n=1}^N$ are generated by λ_t^l. Hence, the null hypothesis is that $\{A_{n+1}^l - A_n^l\}_{n=1}^{N-1}$ are independent and exponentially-distributed with parameter 1.

4.1.2 Thinning process estimation

We estimate parameters of default thinning process z_ρ as following. Let $N_t^{(3,\rho)}$ be the counting process that represents the number of defaults from the portfolio of ρ-rated firms. Then, the intensity process of $N_t^{(3,\rho)}$ is $z_\rho \lambda_t^3$. Here we get, for any s and t $(s < t)$,$\mathrm{E}[N_t^{(3,\rho)} - N_s^{(3,\rho)}] = \int_s^t \mathrm{E}[z_\rho \lambda_u^3]\mathrm{d}u = z_\rho \int_s^t \mathrm{E}[\lambda_u^3]\mathrm{d}u = z_\rho \mathrm{E}[N_t^3 - N_s^3]$. From this relationship between the expectation of $N_t^{(3,\rho)}$ and the expectation of the economy-wide default counting process N_t^3, we obtain

$$z_\rho = \mathrm{E}[N_t^{(3,\rho)} - N_s^{(3,\rho)}]/\mathrm{E}[N_t^3 - N_s^3]. \tag{15}$$

We estimate z_ρ with (15). In addition, we estimate z_m^1 and z_m^2 in the same way.

4.1.3 Estimation Result

The data for parameter estimation are the records on rating changes of Japanese firms. The ratings are announced by Rating and Investment Information, Inc. (R& I). R& I is one of the largest rating agencies in Japan. Since there are no defaults in the data, we treated the rating below BBB- as quasi-default. Then, rating considered are AAA, AA+, AA, AA-, A+, A, A-, BBB+, BBB, BBB-, Default, that is $K = 10$.

We used the rating change records from April 1, 2004 to April 1 2009 for estimating intensity models. During the period (April 1, 2004 — April 1, 2009), 347 up-grades, 125 down-grades and 18 quasi-defaults were observed. Excluding no-business days, we transformed calendar times 2004/4/1, 2005/4/1, $\cdots$ to $t = 0, 1, \cdots$. There are a lot of events are in the same day, so we slide the event times with uniform random number so as to make every event times different.

We execute the maximum likelihood estimation of the parameters with the free statistical software package `R`. In particular, we use the intrinsic function “`optim`” to maximize the objective function. We execute the maximization for 30 sets of initial values, and finally choose the estimates that maximize the objective function among the initial value sets. In addition, we execute Kolmogorov-Smirnov test with `R`, using the intrinsic function “`ks.test`”.

Table 1 shows the estimation result of the intensity processes. As all the obtained p-values are over 0.05, the estimated models are not rejected with the 5% significant level.

Table. 1 Estimated parameters of the intensity processes

	κ	c	δ	γ	λ_0	p-value
up-grade ($l = 1$)	1.745	0.350	1.2(fixed)	90.804	26.486	0.062
down-grade ($l = 2$)	1.643	0.281	1.2(fixed)	168.839	82.676	0.063
default ($l = 3$)	3.450	0.503	1.2(fixed)	23.384	1.181	0.974

We used the records from April 1, 1998 to April 1 2009 on rating changes for estimating thinning process models. Table 2, 3 shows estimated parameters of the thinning processes.

Table. 2 Estimated parameters of the thinning processes (10) and (11)

m	1	2	3	4	5	6	7	8	9
z^1_m	0.9024	0.0732	0.0244	0	0	0	0	0	0
z^2_m	0.8618	0.1290	0.0046	0.0046	0	0	0	0	0

Table. 3 Estimated parameters of the thinning process (12)

ρ	1	2	3	4	5	6	7	8	9	10
z_ρ	0	0	0	0	0	0	0	0.03	0.26	0.71

4.2 Risk analysis of CDOs and CDO-squareds

In this subsection, we show some results of a numerical experiment on a risk analysis of CDOs and a CDO-squared. The purpose of our numerical experiments is just to show some features of our model and to illustrate what we can with our model on a risk analysis of CDOs and CDO-squareds. With our model, we can estimate the exposure of tranches of several CDOs, as our model treats several portfolios simultaneously.

The CDOs in our numerical experiment are collateralized bond obligations (CBOs). A CBO is a CDO with a reference portfolio of corporate bonds. We consider three CDOs (CDO No.1, No.2 and No.3) with different reference portfolios. The reference portfolios in our numerical experiments are described in Table 4. Table 4 shows there are 100 corporate bonds for each rating. The portfolio 1 is high credit quality with many high rating bonds, the portfolio 3 is low credit quality with many low rating bonds, and the portfolio 2 is the average. The reference portfolio of CDO No.1 is the portfolio 1, CDO No.2 is the portfolio 2, and so on. Our settings of CDOs are following:

- Reference portfolio

  - Consist of 300 fixed rate bonds (Maturity: 5 years. Coupon rate: 4%).
- CDO
  - Tranching: Equity [0%, 5%], junior mezzanine [5%, 10%], senior mezzanine [10%, 15%], senior [15%, 85%]
  - Coupon: junior mezzanine 2.5%, senior mezzanine 2.0%, senior 1.0%
  - Equity receives the residual of all coupons from the reference portfolio after superior tranches have received their coupons.

Table. 4 Rating distributions of the portfolios

rating	1	2	3	4	5	6	7	8	9	10
portfolio 1	50	50	50	50	50	10	10	10	10	10
portfolio 2	30	30	30	30	30	30	30	30	30	30
portfolio 3	10	10	10	10	10	50	50	50	50	50
residual	10	10	10	10	10	10	10	10	10	10

Furthermore, we considered the CDO-squared of which reference portfolio consists of junior mezzanines of CDO No.1, 2, 3. The tranches setting of the CDO-squared is the same as CDO No.1, 2, 3.

For setting the model parameters, we used the estimated parameter values in Table1, 2 and 3.

The results of our numerical experiments are the following. Figure 3 shows the loss rate distribution of the tranches of the CDO No.1 and Table 5 shows the values of risk measures, such as value at risk (VaR) and expected shortfall (ES), of the tranches of the CDO No.1. In turn, Figure 4 shows the loss rate distribution of the tranches of the CDO No.2 and Table 6 shows the values of risk measures of the tranches of the CDO No.2. Comparing Figure 4 with Figure 3, or comparing Table 6 with Table 5, the tranches of CDO No.2 are exposed much more risk than CDO No.1. This is because the default intensity of a low credit quality portfolio tends to become higher than that of a high credit quality portfolio, with our thinning processes model.

Table. 5 Risk measures on loss of CDO(No.1) tranches

	Average loss	99% VaR	99% ES	Maximum loss
Portfolio 1	0.52%	2.85%	4.57%	9.74%
Equity	3.50%	19.26%	30.07%	63.36%
Junior Mezzanine	0.04%	0.00%	4.17%	42.52%
Senior Mezzanine	0.00%	0.00%	0.00%	0.00%
Senior	0.00%	0.00%	0.00%	0.00%

As our model treats several portfolios simultaneously, we obtain conditional loss distributions of CDO tranches, on the condition of alternative CDO tranches loss.

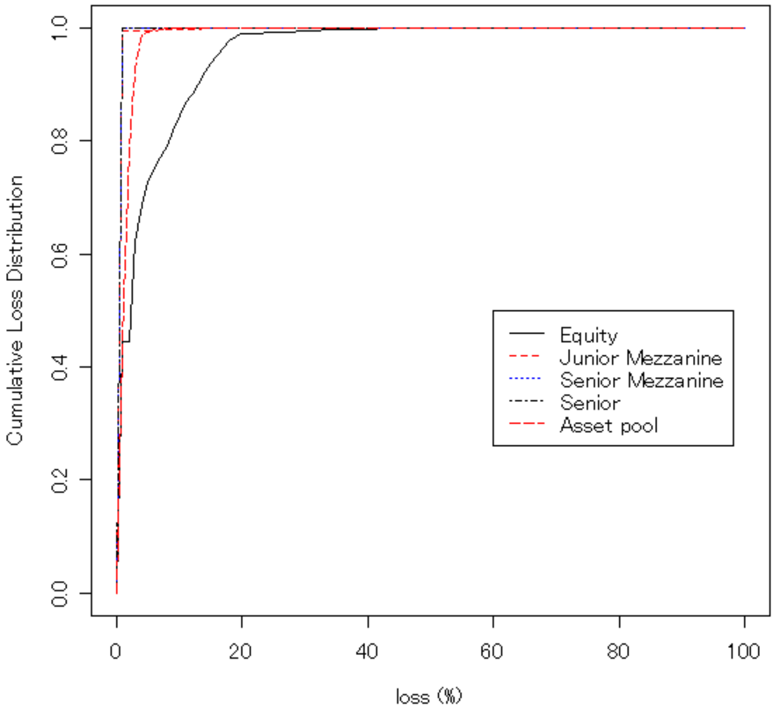

Fig. 3 Tranche loss distributions of CDO(No.1)

Table. 6 Risk measures on loss of CDO(No.2) tranches

	Average loss	99% VaR	99% ES	Maximum loss
Portfolio 2	2.24%	14.40%	15.44%	19.28%
Equity	11.97%	67.67%	73.51%	92.01%
Junior Mezzanine	13.81%	93.96%	94.65%	97.26%
Senior Mezzanine	2.88%	67.58%	79.95%	95.18%
Senior	0.00%	0.00%	0.07%	1.68%

Figure 5 and table 7 show conditional loss distributions of junior mezzanine of CDO No.2; on the conditions that loss of junior mezzanine of CDO No.3 is under 90% or over 90%. We can recognize the difference of loss amount of CDO No.2 caused by loss of CDO No.3. Figure 5 and table 7 show that we can analyze credit risk contagion among several CDOs with the model.

In addition to analyzing risks of CDOs, we execute risk analysis of CDO-squared. Figure 6 shows the loss distribution of the CDO-squared tranches and table 8 shows the risk measures on the CDO-squared tranches loss.

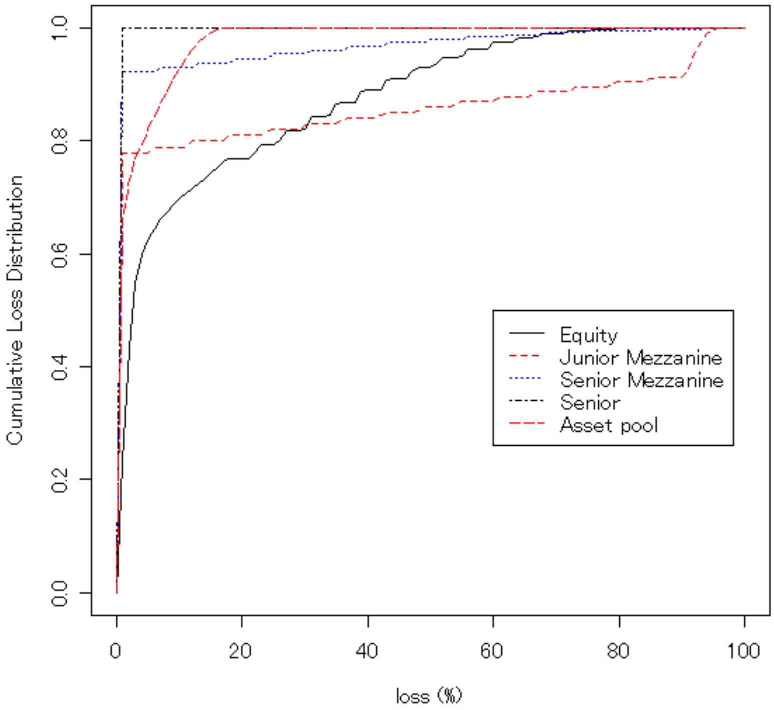

Fig. 4 Tranche loss distribution of CDO (No.2)

Table. 7 Influence of CDO No.3 to CDO No.2 (Risk measure of junior mezzanine of CDO No.2)

	Average loss	99% VaR	99% ES	Maximum loss
No condition	13.81%	93.96%	94.65%	97.26%
Below 90%	0.97%	30.22%	46.85%	90.87%
Over 90%	66.59%	95.11%	95.59%	97.26%

Table. 8 Risk measures on loss of the CDO-squared tranches

	Average loss	99% VaR	99% ES	Maximum loss
Portfolio of JMs of CDOs	12.51%	65.71%	67.00%	78.86%
Equity	15.04%	78.76%	82.44%	93.05%
Junior Mezzanine	26.51%	96.67%	97.18%	98.89%
Senior Mezzanine	25.72%	97.26%	97.59%	99.09%
Senior	11.30%	61.98%	63.51%	78.02%

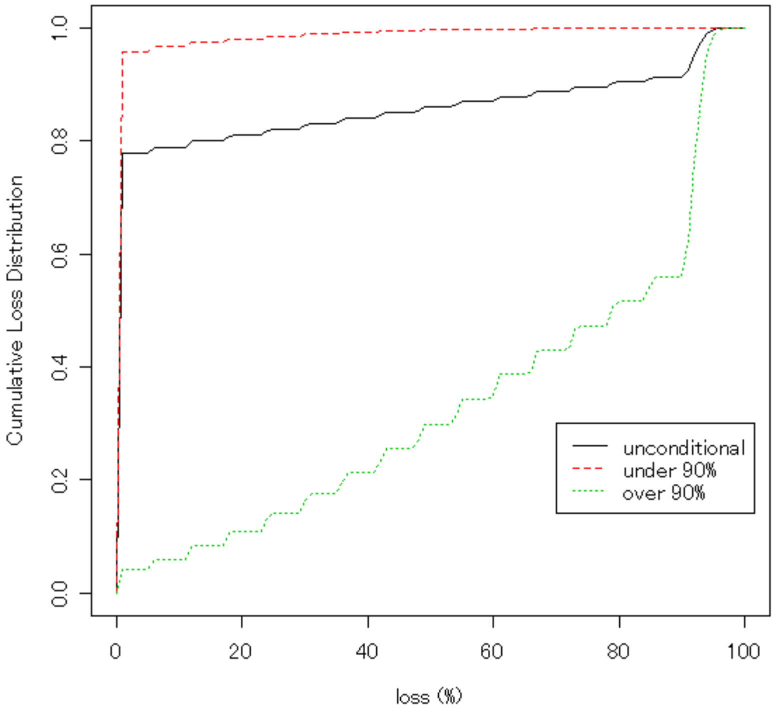

Fig. 5 Influence of CDO No.3 to CDO No.2 (Conditional loss distribution of Junior Mezzanine of CDO No.2)

At the end of this section, we show a result of a parameter sensitivity analysis of CDO-squared tranches loss. Table 9 shows sensitivity of the risk measures of junior mezzanine loss of the CDO-squared to κ^3, the decline speed parameter of the default intensity λ_t^3. From the results in table 9, we recognize that as the κ^3 decreases, the default risk increases.

Table. 9 Parameter sensitivity of Risk measures of the CDO-squared tranche

κ^3	Average loss	99% VaR	99% ES	Maximum loss
2.45 (−1.00%)	28.71% (+11.10%)	95.74% (+0.81%)	96.26% (+0.79%)	98.26% (+0.37%)
3.45 (±0.00%)	17.60% (±0.00%)	94.93% (±0.00%)	95.47% (±0.00%)	97.89% (±0.00%)
4.45 (+1.00%)	11.40% (−6.20%)	93.93% (−1.00%)	94.77% (−0.70%)	97.56% (−0.33%)

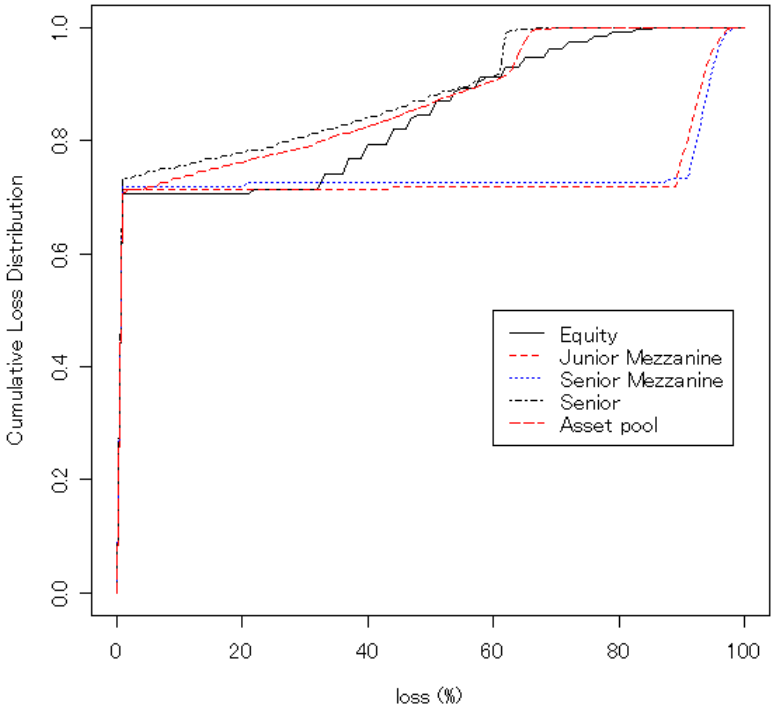

Fig. 6 Tranche loss distributions of CDO-squared

5 Concluding remarks

We presented a new model of credit events based on the top-down approach. Here we were interested in modeling rating changes (up-grades and down-grades) and defaults in the whole economy. Our model consists of the intensities of some credit events that are specified by the self-exciting processes with a state-dependent property. The model is similar to that of Giesecke and Kim [4], but different from them since we suppose the jump sizes of our intensity model have an upper limit.

In addition, we specify the random thinning processes in terms of the distribution of credit ratings in the sub-portfolios. Such a specification of the random thinning can ensure that the credit risk of one sub-portfolio increases as the whole credit quality in the portfolio becomes worse. Also, our model can capture credit risk contagion among several portfolios so that it is possible to evaluate the portfolios simultaneously.

In addition, we present the credit event simulation algorithm based on the model. As we illustrate some numerical examples on risk analysis of CDO and a CDO-squared, our model can be worked effectively to analyze risks of portfolio credit derivatives.

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