USING LOCAL CORRELATION MODELS TO IMPROVE OPTION HEDGING

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Overview: Correlation plays a tremendous role when pricing and hedging multi-asset derivative products. In this article we show how local correlation

- Accommodates a basket skew with the local volatility dynamics of its constituents,
- Smoothes the risk-management of a whole class of equity products.

More importantly, stochastic correlation is made more accessible in production for daily pricing and risk management. We provide a detailed description of the pricing methodology and implementation.

This is an incremental improvement of the standard multi-dimensional local volatility model with deterministic correlation.

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MOTIVATION

In the local volatility framework, carry drives daily profits & losses on delta-neutral portfolios $\pi - \sum_{i=1}^{n} \frac{\partial \pi}{\partial S_i} S^i$:

$$\Delta PL_t \approx \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 \pi}{\partial S_i S_j} S_t^i S_t^j \left(\frac{\Delta S_t^i}{S_t^i} \frac{\Delta S_t^j}{S_t^j} - \rho_t^{ij} \sigma_i \left(t, S_t^i\right) \sigma_j \left(t, S_t^j\right) \Delta t \right)$$

With π, the value-function of a multi-underlying option. Implied correlation is often a straight time-dependent function $\left(\rho_t\right)_{i,j}$ either:

- Calibrated on the term-structure of some at-the-money basket implied volatility term-structure,

- Statistically estimated from historical data.

Under stressed market conditions, realized covariance might exceed what was initially expected:

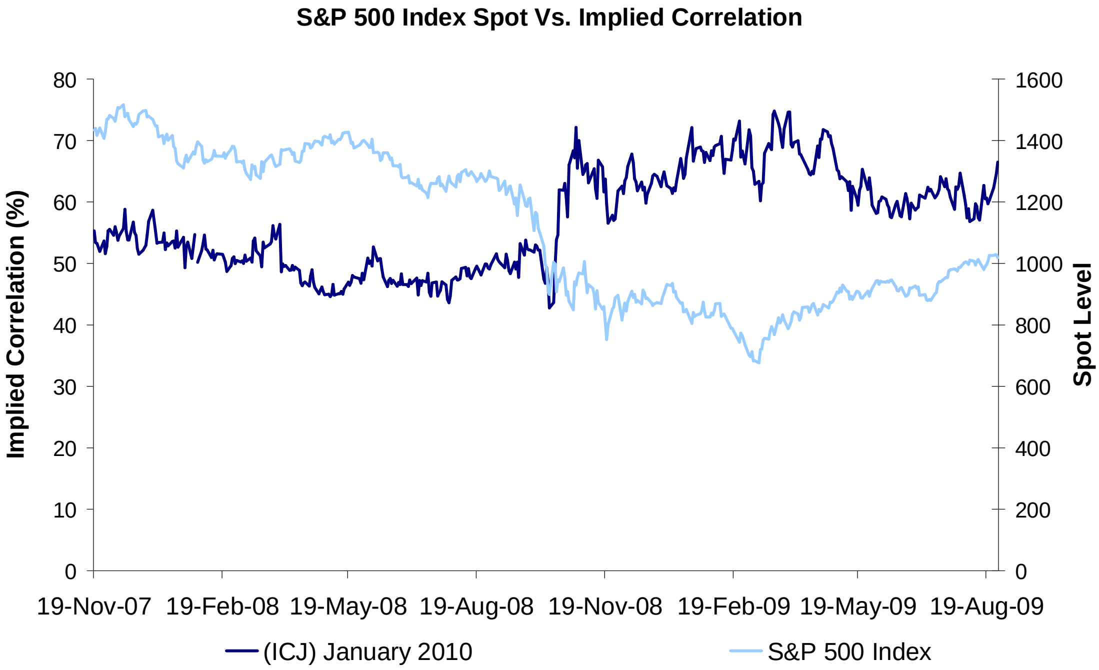
$$\frac{\Delta S_t^i}{S_t^i} \frac{\Delta S_t^j}{S_t^j} > \rho_t^{ij} \sigma_i \left(t, S_t^i\right) \sigma_j \left(t, S_t^j\right) \Delta t$$

Such a scenario would severely damage the risk-management of trading books embedding short correlation exposure. Assuming well-marked local volatilities, a trader needs to shift correlation upward by the shift matrix $\Delta \rho$ so that under the risk-neutral measure:

$$E_t \left[\frac{\Delta S_t^i}{S_t^i} \frac{\Delta S_t^j}{S_t^j} \right] = \left(\rho + \Delta \rho\right)_t^{ij} \sigma_i \left(t, S_t^i\right) \sigma_j \left(t, S_t^j\right) \Delta t$$

Incurring a brutal $\frac{\partial \pi}{\partial \rho} \cdot \Delta \rho$-loss to stop the bleeding.

According to historical correlation measures (ICJ) implied from past prices of S&P 500 index option expiring on January 2010, implied correlation is extremely volatile:



Incorporating correlation risk in pricing models is therefore highly recommended for pricing and hedging multi-underlying options.

However, implementing stochastic correlation is complicated in practice.

We can summarize the difficulties as follows:

- Ill-posed calibration problem with very few market information compared to the number of parameters to determine;
- Time-consuming diffusion scheme given that a matrix factorization is required at each time step of the discretization grid.

This paper addresses these issues and proposes a **mathematical model** as well as **an efficient algorithm to calibrate** and **run the local correlation model** in production.

LOCAL CORRELATION MODEL

In what follows, C denotes the convex cone of symmetric positive semi-definite matrices with a diagonal equals to [1,…1] whereas W^A stands for a multi-dimensional Brownian motion with instantaneous correlation A. In particular, W^Θ is a single Brownian motion given that Θ defined by:

$$\Theta := \begin{bmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{bmatrix}$$

Admits only one unique eigenvector [1,…,1].

The **driving philosophy** behind equity derivatives modelling is **to transfer as much hedge as possible on underlying stocks**. It is one of the reasons of the success of local volatility models for years. Indeed, delta partially grasps the vega exposure so that only the residual vega is hedged through options or variance swaps.
The local correlation model relies on this principle whereby part of the correlation exposure is transferred on delta. We dynamically accompany the expected correlation change with respect to market condition which enables to smooth Pnl.

Without loss of generality, we omit terms corresponding to financing, repos and dividends. The dynamic is written as follows:

$$\begin{cases} \frac{dS_t^1}{S_t^1} & = & \sigma_1\left(t, S_t^1\right) dW_t^{\rho(\lambda),1} \\ \vdots & \vdots & \vdots \\ \frac{dS_t^n}{S_t^n} & = & \sigma_n\left(t, S_t^n\right) dW_t^{\rho(\lambda),n} \end{cases}$$

Where $W^{\rho(\lambda)} = [W^{\rho(\lambda),1},\ldots, W^{\rho(\lambda),n}]$ is a multi-dimensional Brownian motion with stochastic correlation:

$$\rho(\lambda)_t := \left(1 - \lambda\left(t, S_t^1, \ldots, S_t^n\right)\right)\rho + \lambda\left(t, S_t^1, \ldots, S_t^n\right)\Theta$$

ρ belongs to C as well as Θ so that ρ(λ) remains in C for any value λ in [0,1].

The local lambda function λ is fine-tuned according to payoff and available information:

1. **Implied Approach - Mostly appropriate for world indices or baskets of very liquid stocks**
   Basket option market quotes on various strikes and maturity provide first-hand information about implied correlation. Therefore, we opt for a model calibration

to market observables by building a simplified local lambda function $\lambda\left(t,\sum_{i=1}^{n} w_i \frac{S_t^i}{S_0^i}\right)$ fitting the basket implied volatility surface.

2. **Direct Approach – Mostly appropriate for regular stock baskets**
Suppose basket options are not liquid enough to extract relevant information from the market. We can still analyze the statistical distributions of the correlation with respect to the basket components as a whole or in some directions to infer a judicious functional form that mimics historical behaviours. For instance, on rainbow options, we suggest:

$$\lambda\left(t,S_t^1,\ldots,S_t^n\right)=\lambda_0 f\left(t,L_t\right)$$

With λ_0 in [0,1] and L defined by

$$L_t=\sum_{i=1}^{n} w_{(i)} \ln\left(\frac{S_t^{(i)}}{S_0^{(i)}}\right)$$

Where $S_{(i)}$ is the i^{th} best performing asset of the basket.

You can think of λ_0 as a bump that the function f deforms depending on current market conditions L_t. Notice that f also depends on the time t so that short-term and long-term correlation dynamics can differently respond to market moves. This is of particular relevance when estimation follows abnormal market period as from September 08 to March 09. The model embeds a view on future correlation according to market regimes.

In both cases local correlation adjusts S_i-delta by $\frac{\partial\pi}{\partial\lambda}\frac{\partial\lambda}{\partial S_i}$.

A naïve approach to diffuse a basket of n spots under a local correlation scheme consists in performing successive Choleski decompositions - with $O(n^3)$ complexity - at each time step of the Euler discretization grid according to current time and spot levels. This is extremely time-consuming: Neither a price nor all necessary risk measures can be produced in a reasonable amount of time.

A dimension extension similar to what was introduced in [1] helps to by-pass this limitation.

We easily remark that $W^{\rho(\lambda)}$ is the weighted-sum of two independent n-dimensional Brownian motions W^ρ and W^Θ:

$$dW_t^{\rho(\lambda)} = \sqrt{1-\lambda\left(t,S_t^1,...,S_t^n\right)}dW_t^{\rho} + \sqrt{\lambda\left(t,S_t^1,...,S_t^n\right)}dW_t^{\Theta}$$

Given that W^{Θ} degenerates into a single Brownian motion, we only need to factorize ρ to build Brownian increments with instantaneous correlation $\rho(\lambda)$.

More generally, assume λ in $[\lambda_{\min},1]$ included in $[-1,1]$, then

$$dW_t^{\rho(\lambda)} = \sqrt{1-\frac{\lambda\left(t,S_t^1,...,S_t^n\right)-\lambda_{\min}}{1-\lambda_{\min}}}dW_t^{(1-\lambda_{\min})\rho+\lambda_{\min}\Theta} + \sqrt{\frac{\lambda\left(t,S_t^1,...,S_t^n\right)-\lambda_{\min}}{1-\lambda_{\min}}}dW_t^{\Theta}$$

And the matrix to decompose becomes $(1-\lambda_{\min})\rho+\lambda_{\min}\Theta$.

This algorithm has the same complexity as the classic multi dimensional local volatility model with only one Choleski decomposition to perform before starting the Monte Carlo simulations. Here, the innovation is that an extra independent Brownian motion is simulated to generate Brownian increments under the local correlation framework at each node of the path.

MODEL CALIBRATION USING FIXED-POINT ALGORITHM

Basket options with payoff $\left(\sum_{i=1}^{n} w_i \frac{S_T^i}{S_0^i} - K\right)^+$ are mainly sensitive to global correlation shift so that we search for a local lambda function which depends on stock spots through basket level $B_t = \sum_{i=1}^{n} w_i \frac{S_t^i}{S_0^i}$

In the presence of a set of basket option quotes for various strikes and maturities, the functional form λ is calibrated with the **fixed-point algorithm**.

Given a local surface λ, we evaluate basket options on various maturity/strike buckets (T,K) in order to imply the Black-Scholes volatility surface $\Phi(\lambda)$ generated by the local correlation model.

The contracting mapping is then the application which turns the local lambda surface λ into a basket implied volatility surface $\Phi(\lambda)$:

$$\lambda(*,\bullet) \to \Phi(\lambda)(*,\bullet)$$

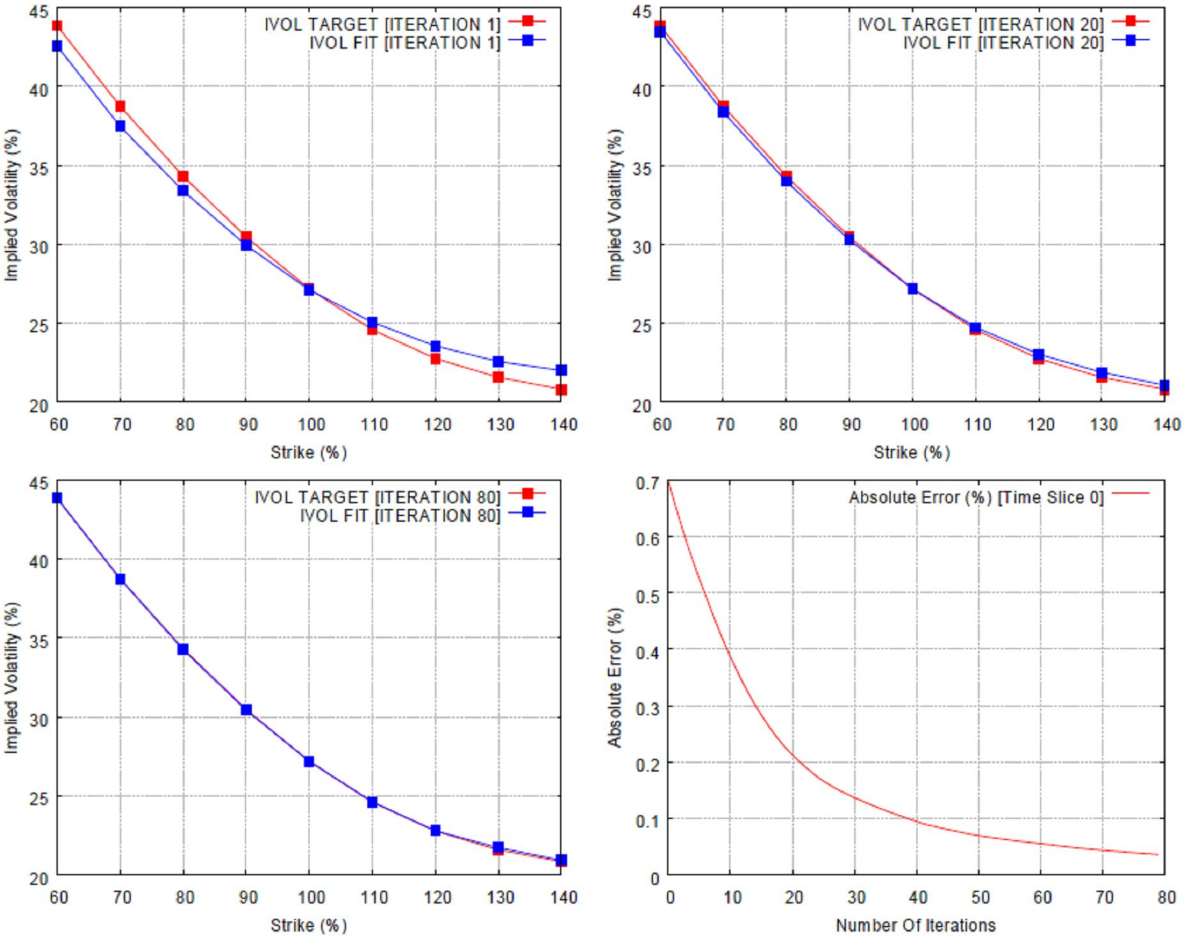
We proceed as with the local volatility calibration previously discussed in [2][3]:

- Start with a relevant initial guess λ_0;
- Price basket options in the local correlation / volatility framework;
- Imply volatilities to obtain the mapping $\Phi(\lambda)$;
- Repeat the following iterative scheme until convergence holds

  $$\lambda_{i+1}(*,\bullet)=\lambda_i(*,\bullet)+\ln\left(\frac{\Sigma_{\text{Market}}(*,\bullet)}{\Phi(\lambda_i)(*,\bullet)}\right)$$

  Σ_{Market} denotes the Black-Scholes volatility surface implied from quoted basket options in the market.

We run the algorithm on an equi-weighted basket composed of eight stock market indices. For a given expiry the iterative steps to build a functional form λ so that the local correlation model fit a “fictive” market basket skew - in red - are displayed below.



We compare valuation and Greeks on a 10MUSD short position on 1Y best-of put options priced under the local correlation / volatility model (LVLC) calibrated on the red curve against the standard local volatility model with constant correlation matching ATM 1Y basket options (LV).

Results are expressed in KUSD:

Model	PV	Delta Cash	Gamma Cash	Theta
LV	-487	2910	362	0,7
LVLC	-648	4802	709	2,2

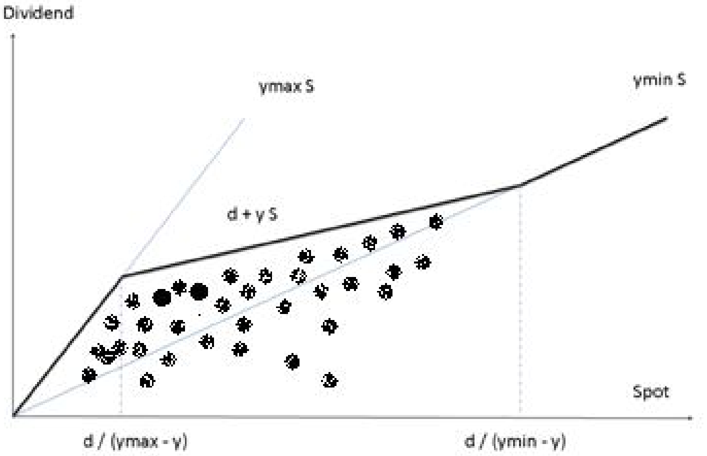
In terms of valuation impact with respect to tenor, it gives (in KUSD):

Model \ Tenor	1Y	3Y	5Y
LV	-487	-820	-957

LVLC	-648	-1066	-1211

MODEL ESTIMATION USING ENVELOP APPROACH

The **Envelop modeling approach** models the upper (respectively lower) bound of a random quantity Y with a hyperplane X.
This technique is successfully used to model dividends with respect to spot:



The upper dividend Envelop model is then given by the equation:

$$D(S)=\min\left(y_{\max}S,\max\left(y_{\min}S,d+yS\right)\right)$$

We rely on a comparable approach to define the local lambda function λ by estimating a conservative - according to the correlation sensitivity of the derivatives product under consideration - correlation envelop depending on spot levels.

In the stock world, the market does not provide sufficient information to calibrate basket skews – no or few basket option quotes available -. Besides more exotic pay-offs can be highly sensitive to twisted deformation of the correlation matrix. This “**Chewing-Gum**” effect fully detailed in [4] which occurs when spot dispersion is such that the average basket level barely changes whereas worst- and/or best-of performances are deeply altered -cannot be wisely considered with correlation shifts λ only controlled by basket level B.

We therefore proceed with an empirical analysis depending on the multi-underlying option to evaluate. In this regard, we consider a pay-off which pays at maturity T a put on the best performance of a basket made up of n assets:

$$\left(K-\max\left(\frac{S_T^1}{S_0^1},\dots,\frac{S_T^n}{S_0^n}\right)\right)^+$$

Where K stands for the option's strike expressed in %. It brings dependence on the best performing asset so that it seems coherent to investigate a spot local correlation which singles out the best performance. Furthermore, the sensitivity with respect to correlation is positive so that these options are traded at the "ask" implied correlation level of the seller.

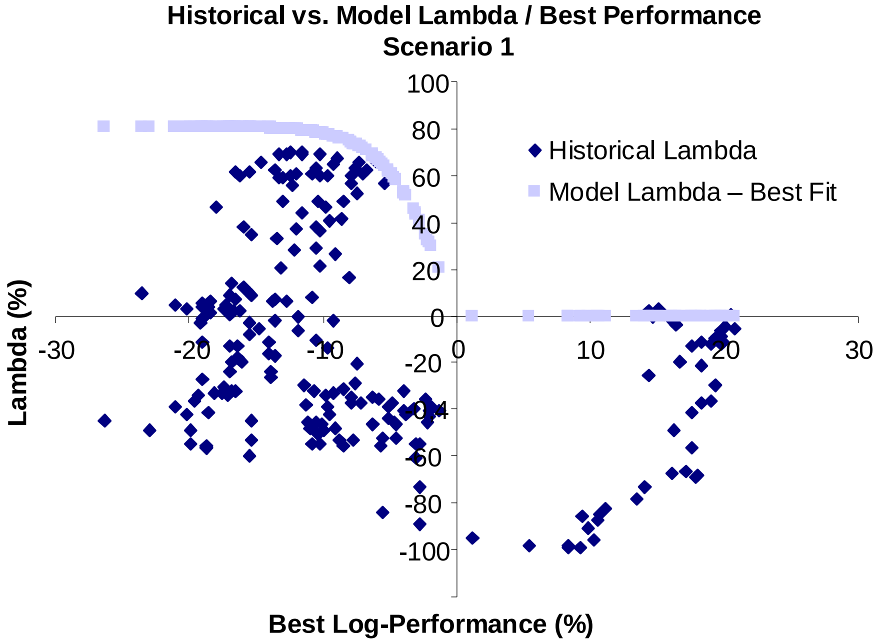
Consider now the following parameterization – a standard Fisher transformation:

$$\lambda\left(S_t^1,\dots\dots,S_t^n\right)=\max\left(-\lambda_0\times\tanh\left(\frac{s}{\lambda_0}L_t\right),\lambda_{\min}\right)$$

Where, $L_t=\max_{i=1,\cdots,n}\left[\log\left(\frac{S_t^i}{S_0^i}\right)\right]$.

Toying with parameters λ_0, λ_{min} and s, we model λ from worst-case scenarios according to our risk-aversion by drawing more or less conservative correlation envelopes:

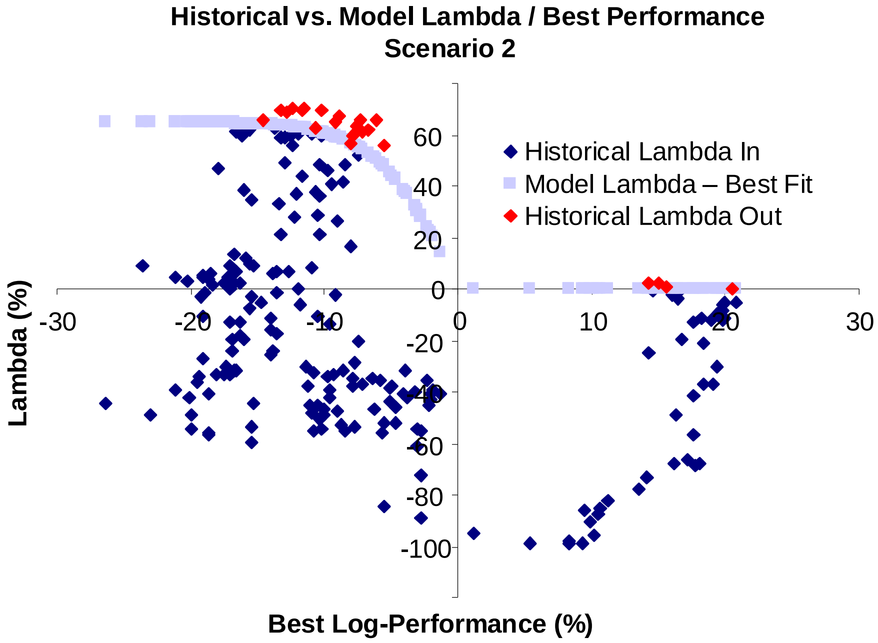
> From last year's log-performance data (2008-2009), we compute λ_{Histo} and display its relation to the best performer. Best-of put options are sensitive to negative log-performances so that the analysis focuses on downside correlation.
>
> **Scenario 1. Conservative business level**
>
> Model correlation envelops historical realizations of λ_{Histo} in such a way that the empirical probability that realized correlation exceeds model correlation is null:
>
> $$\lambda_0=81\%,\ \lambda_{\min}=0\%,\ s=16\Rightarrow P\left\{\lambda_{\text{Histo}}<\lambda_{\text{Model}}\,\middle|\,L_t<0\right\}=100\%$$



Scenario 2. Moderate business level

Historical correlation outperforms model correlation on some rare events but it remains below most of the time:

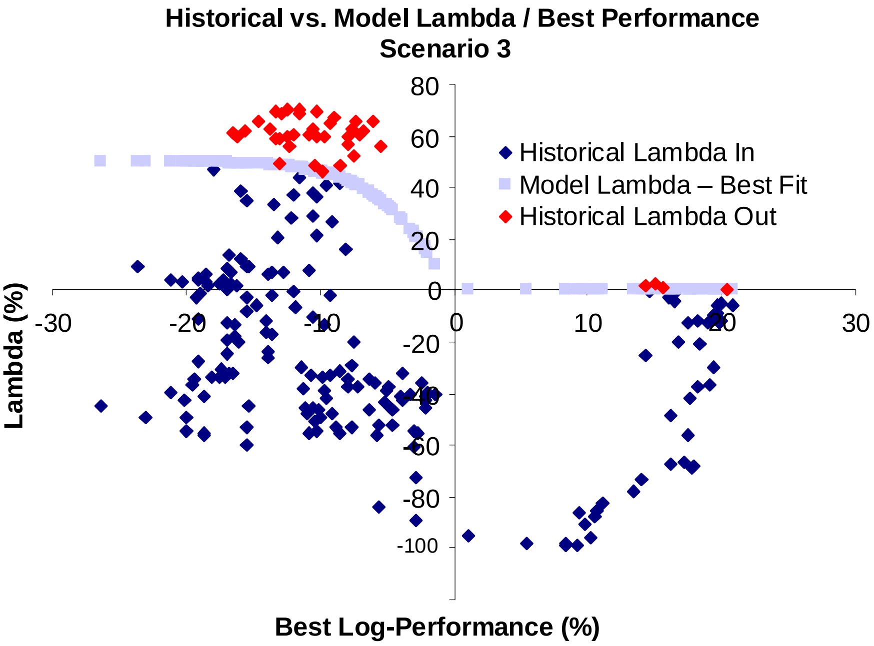
$$\lambda_0 = 65\%,\ \lambda_{\min} = 0\%,\ s = 11 \Rightarrow P\left\{\lambda_{\text{Histo}} < \lambda_{\text{Model}} \,\middle|\, L_t < 0\right\} = 90\%$$



Scenario 3. Acceptance business level

This scenario suits agent with a risk appetite such that model correlation underestimates realized correlation 20% of time:

$$\lambda_0 = 50\%, \lambda_{\min} = 0\%, s = 8 \Rightarrow P\left\{\lambda_{\text{Histo}} < \lambda_{\text{Model}} \middle| L_t < 0\right\} = 80\%$$



The functional form λ might not be appropriate to model mean correlation level. However it perfectly fits the envelope which is enough to conservatively hedge even in tough times.

We focus on a 10MEur short position on Best-Of put. We compare the valuation impact with model choice between:

- Black - Scholes Model (BS)
- Local Volatility Model (LV)
- Local Correlation / Volatility Model (LVLC).

We stress the portfolio under three **daily global spot shift scenarios** at respectively -10%, 0% and +10% from strike levels keeping volatilities and correlations still. The 10% absolute daily shift is an extreme scenario which was encountered few times during the crisis from September 2008 until March 2009.

Suppose gamma PL is locked every 1%-spot shift by rebalancing the delta position by GAMMA_CASH(0), the gamma cash amount computed with 0%-spot shift. Then:

$$\mathrm{PL\_HEDGE} = -\mathrm{DELTA\_CASH}(0) * \mathrm{SPOT\_SHIFT} - \mathrm{SGN}(\mathrm{SPOT\_SHIFT}) * \mathrm{GAMMA\_CASH}(0) * \sum_{i=\mathrm{SGN}(\mathrm{SPOT\_SHIFT})*1\%}^{\mathrm{SPOT\_SHIFT}} \mathrm{SPOT\_SHIFT} - i$$

We compare this expression to PL_OPTION= PV(SPOT_SHIFT)-PV(0) in order to evaluate the robustness of the pricing models.

Results are displayed in KEur.

SPOT_SHIFT	MODEL	PV	DELTA_CASH	GAMMA_CASH	VEGA	THETA	PL_OPTION	PL_HEDGE	PL
-10	BS	-881	4 363	-155	-11	0.6	-359	391	32
	LV	-765	4 020	-167	-9	0.3	-324	345	21
	LVLC [Sc.1]	-1 099	5 857	-45	-13	1.1	-497	598	101
	LVLC [Sc.2]	-989	5 342	-107	-15	0.6	-437	464	27
	LVLC [Sc.3]	-912	4 889	-187	-13	0.6	-396	433	37
0	BS	-522	3 144	-170	-15	0.9			
	LV	-440	2 695	-167	-15	0			
	LVLC [Sc.1]	-603	4 117	-413	-22	0.3			
	LVLC [Sc.2]	-552	3 604	-231	-19	0.1			
	LVLC [Sc.3]	-516	3 275	-234	-19	0.1			
10	BS	-281	1 946	-132	-15	0.9	241	-230	11
	LV	-240	1 522	-109	-12	0.7	200	-186	14
	LVLC [Sc.1]	-317	2 066	-239	-12	0.8	285	-205	80
	LVLC [Sc.2]	-295	1 906	-160	-12	0.8	257	-245	12
	LVLC [Sc.3]	-279	1 798	-164	-12	0.9	237	-210	27

The short Best-Of Put position is long gamma so PL=PL_OPTION+PL_HEDGE is always positive. However, **following a 10%-spot shift, we can expect some costly correlation remarking in BS and LV** whereas LVLC has already anticipated the correlation change – it is all priced in!

CONCLUSION

Following the equity correlation debacle of 2008, local correlation models need from now on to be promoted and carried out in production to price and risk-manage multi-asset derivative products. Basket option prices permit to perform robust calibration. Otherwise parametric estimation tailored to payoffs under consideration becomes relevant. A correlation envelope modelling in line with the business risk appetite is then proposed. A smart implementation using dimension extension enables to obtain at no extra cost a much richer model up and running in information systems. In particular, Greeks are adjusted to reflect a part of the correlation risk reallocation from cega to delta whereas the gamma-theta ratio is substantially improved.

THANKS

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