Multivariate Stress Testing for Solvency II

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In Favour of Probabilistic Stress Testing

- Stress testing consists in “choosing scenarios that are costly and rare, and then putting them to the valuation model” [Berkowitz, 2004].
- Rare scenarios that have no assessment of likelihood attached have limited value - can always find a “ruin scenario”, but is it plausible?
- Stress scenarios are best formalised as tail events from probability distributions.
- But which distribution? The multivariate distribution of the factors causing stress events? The univariate distribution expressing the effect of the stress event to the bottom line?

FSA Proposes Reverse Stress Test

We are proposing to introduce a 'reverse-stress test' requirement, which would apply to banks, building societies, CRD investment firms and insurers, and would require firms to consider the scenarios most likely to cause their current business model to become unviable.

http://www.fsa.gov.uk/pubs/cp/cp08_24_newsletter.pdf

As Explained by FSA

Reverse-stress testing of a firm's business-model vulnerabilities

> *3.12 We are proposing to introduce to our Handbook a new requirement for a firm explicitly to identify and assess the scenarios most likely to cause its current business plan to become unviable. In this context, a firm's business plan should be assumed unviable at the point that crystallising risks cause the market to lose confidence in it, with the consequence that counterparties and other stakeholders are unwilling to transact with it or provide capital to the firm and, where relevant, that existing counterparties may seek to terminate their contracts. Recent experience suggests that such a point may be reached well before a firm's regulatory capital is exhausted.*

As Explained by FSA (II)

> *3.13 The intention behind the introduction of this new requirement is to encourage firms: first, to explore more fully the vulnerabilities of their current business plan (including 'tail risks' as well as milder adverse scenarios); second, to make decisions that better integrate business and capital planning; and third, to improve their contingency planning. In the case of 'common platform firms' the reverse-stress test requirement elaborates the MiFID-derived requirements on risk control in SYSC7.1.2R to 7.1.7R. The proposed requirement is intended to be holistic, so firms should consider liquidity risks as well as risks to their capital positions.*

(CP08/24) http://www.fsa.gov.uk/pubs/cp/cp08_24.pdf

One-Period Risk Model

Consider a one-period risk model for an enterprise taking the form

$$V = g\begin{pmatrix} X_1 \\ \vdots \\ \vdots \\ X_d \end{pmatrix},$$

or, more compactly, $V = g(\mathbf{X})$, where

- the random vector $\mathbf{X} \in \mathbb{R}^d$ describes changes in fundamental risk drivers/factors such as equity prices, foreign exchange rates, term structures of interest rates and credit spreads;
- $g : \mathbb{R}^d \to \mathbb{R}$ is the risk impact function;
- V is the net asset value at the end of the time period;
- $v_0 = g(\mathbf{0})$ is the initial net asset value.

Stress Testing

1. Consider the distribution of the risk factors $\mathbf{X}$
2. Define set of plausible risk factor scenarios with reference to distribution
3. Identify the most destructive scenario(s) in this set with respect to the risk impact function g.

One attempt to formalise this set of steps has resulted in the concept of the LSLE, or least solvent likely event.

Reverse Stress Testing

1. Identify the scenarios for $\mathbf{X}$ that lead to ruin ($V \leq 0$).
2. Attempt to infer the most likely scenario among the ruin scenarios.

One attempt to formalise this set of steps has resulted in the concept of the MLRE, or most likely ruin event.

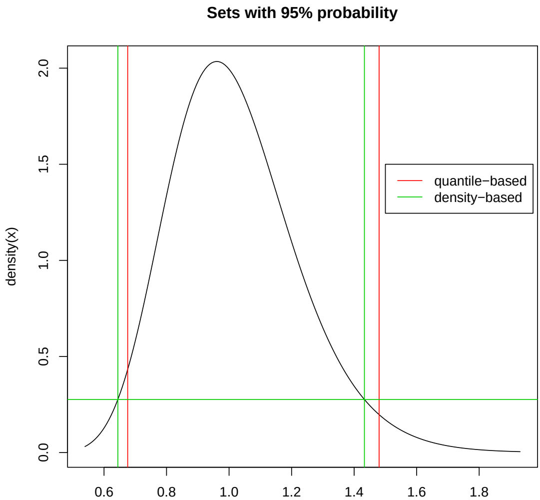
2 Theory

- Model Set-Up
- What are Plausible/Likely Scenarios?
- LSLEs and MLREs
- Convex Analysis and the LSLE

In One Dimension

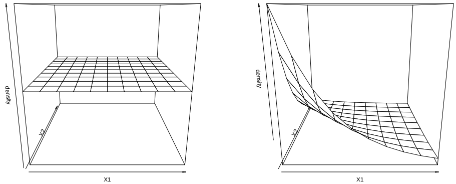
- For a single risk factor X we can use an inter-quantile range to define a set of plausible scenarios, particularly when X has a well-behaved unimodal distribution.
- For $0 < \theta < 1$ let $q_\theta(X)$ denote the θ-quantile of X. Assume that X has a continuous and strictly increasing distribution function so that $q_\theta(X)$ is always unambiguously defined.
- For any α satisfying $1 > \alpha > 0.5$, the inter-quantile interval $I = [q_{1-\alpha}(X), q_\alpha(X)]$ forms a set satisfying $P(I) = 2\alpha - 1$. For large α it is very likely that X will fall in this range.
- This is not the only interval with probability $2\alpha - 1$. Can also create sets which maximise the minimum density.

An Example


Sets with 95% probability
density(x)
0.0
0.5
1.0
1.5
2.0
0.6
0.8
1.0
1.2
1.4
1.6
1.8
quantile−based
density−based

In Higher Dimensions?

The density-based method would correspond to considering contours of equal density that enclose sets of prescribed probability. But this would only work well for well-behaved unimodal distributions (like normals or ellipticals). Consider a pair of independent uniforms or independent exponentials: how would you define the sets in these cases?



The Concept of Halfspace Depth

For any $\mathbf{y} \in \mathbb{R}^d$ and any directional vector $\mathbf{u} \in \mathbb{R}^d \setminus \{0\}$, consider the closed half space

$$H_{\mathbf{y},\mathbf{u}} = \left\{\mathbf{x} \in \mathbb{R}^d \ : \ \mathbf{u}'\mathbf{x} \leq \mathbf{u}'\mathbf{y}\right\},$$

which is bounded by the hyperplane through $\mathbf{y}$ with normal vector $\mathbf{u}$.
Depth at a point $\mathbf{y}$ is defined by

$$\text{depth}(\mathbf{y}) = \inf_{\mathbf{u}:\mathbf{u}\neq\mathbf{0}} P(H_{\mathbf{y},\mathbf{u}}),$$

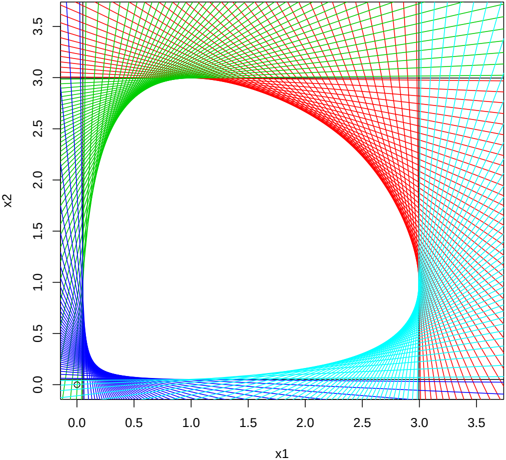
the smallest probability of a closed half space containing $\mathbf{y}$.
For $\alpha > 0.5$ we define the depth set

$$D_\alpha = \left\{\mathbf{y} \in \mathbb{R}^d \ : \ \text{depth}(\mathbf{y}) \geq 1 - \alpha\right\},$$

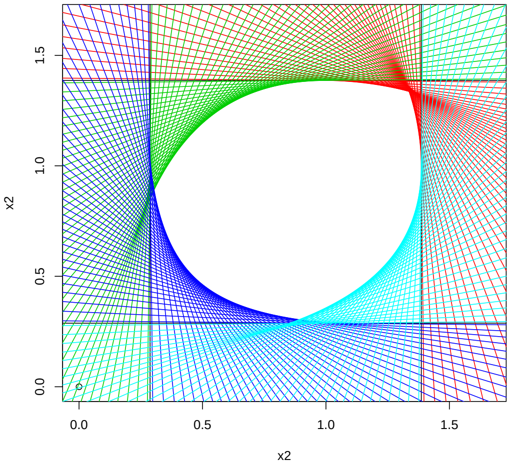
and refer to its boundary as a depth contour. It can be shown that

$$D_\alpha = \bigcap \left\{H_{\mathbf{y},\mathbf{u}} \ : \ P(H_{\mathbf{y},\mathbf{u}}) > \alpha\right\}.$$

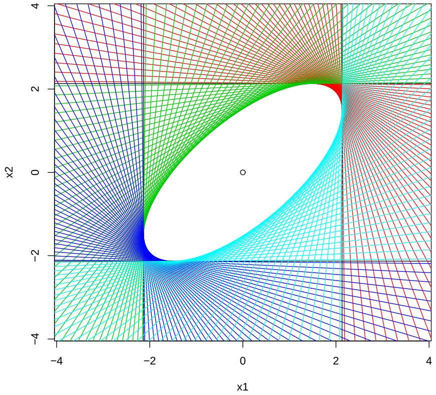
Two Independent Exponentials, $D_{0.95}$



Two Independent Exponentials, $D_{0.75}$



A bivariate Student distribution, $D_{0.95}$



$\nu = 4$, $\rho = 0.7$

Commentary on examples

- Note how the depth set in the exponential case has a smooth boundary for $\alpha = 0.95$ and sharp corners for $\alpha = 0.75$.
- The depth set for an elliptical distribution is an ellipsoid.
- A remarkable property of elliptical distributions: the contours of equal depth and the contours of equal density are ellipsoidal.

Origins of the Concept

Halfspace depth seems to have been first introduced by [Tukey, 1975] as an empirical concept. Given data $X_1, \ldots, X_n$ he measured the depth at $\mathbf{y}$ by

$$\text{depth}(\mathbf{y}) = \min_{\mathbf{u}:\mathbf{u}\neq\mathbf{0}} \frac{\sum_{i=1}^{n} I(X_i \in H_{\mathbf{y},\mathbf{u}})}{n}.$$

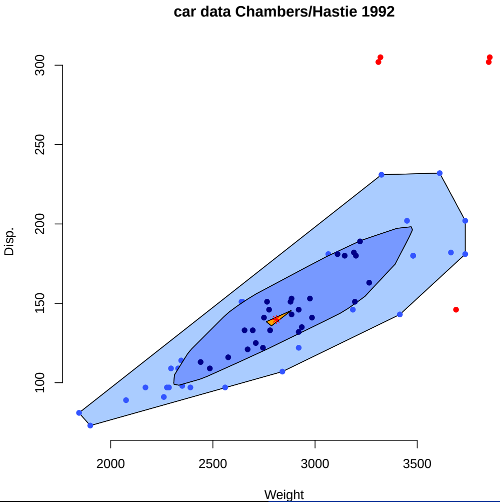
[Rousseeuw et al., 1999] introduced an exploratory graphical tool known as the bagplot. By linking points with similar depth they obtained a kind of multivariate analog of the boxplot.

[Rousseeuw and Ruts, 1999] investigated halfspace depth for probability distributions and gave examples.

[Massé and Theodorescu, 1994] studied similar ideas but defined their sets by

$$\bigcap \left\{ H_{\mathbf{y},\mathbf{u}} \;:\; P(H_{\mathbf{y},\mathbf{u}}) \geq \alpha \right\}.$$

A bagplot


car data Chambers/Hastie 1992
Disp.
100
150
200
250
300
Weight
2000
2500
3000
3500

2 Theory

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- LSLEs and MLREs
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The LSLE

Recall that the net asset value of the company is given by $V = g(\mathbf{X})$. The smaller the value of V, the worse the solvency of the company concerned, and $V \leq 0$ represents ruin.

- We are interested in

$$\mathbf{x}_{\text{LSLE}} = \arg\min \{g(\mathbf{x}) \; : \; \mathbf{x} \in D_\alpha\} \;,$$

  the scenario in the likely set that leads to the worst solvency outcome.

- We could also consider

$$\mathbf{x}_{\text{LSLE}} = \arg\max \{v_0 - g(\mathbf{x}) \; : \; \mathbf{x} \in D_\alpha\} \;,$$

  the scenario maximising the loss.

A Well-Behaved case

For simplicity, and because it leads to insightful results, consider:

1. a linear impact function $g(\mathbf{x}) = \mathbf{u}'\mathbf{x} + v_0$ where $v_0 > 0$;
2. an elliptical distribution for $\mathbf{X}$.

We can show that

$$g(\mathbf{x}_{\text{LSLE}}) = \mathbf{u}'\mathbf{x}_{\text{LSLE}} + v_0 = q_{1-\alpha}(\mathbf{u}'\mathbf{X} + v_0) = q_{1-\alpha}(V)$$

- In words: the LSLE is a scenario that leads exactly to the $(1-\alpha)$-quantile of the net asset value distribution.
- Observe also that $v_0 - g(\mathbf{x}_{\text{LSLE}}) = v_0 - q_{1-\alpha}(V) = q_\alpha(v_0 - V)$, which can be thought of as the α-VaR.

When can we identify VaR-attaining scenarios in general?

Definition of a Dual Problem

An answer to the reverse stress testing problem can be derived by computing:

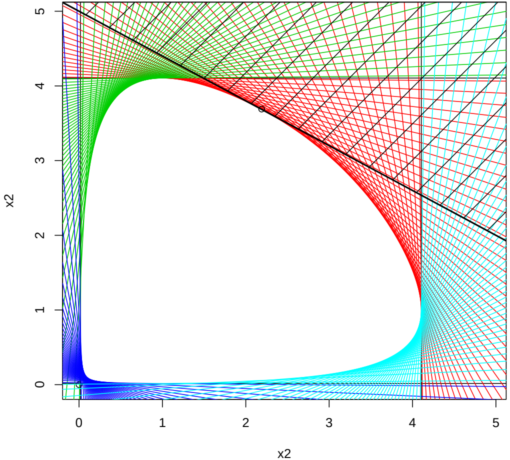
$$\mathbf{x}_{\text{MLRE}} = \arg\max\{\text{depth}(\mathbf{x}) \;:\; g(\mathbf{x}) \leq 0\}\,.$$

Contrast this with the LSLE:

$$\begin{aligned}\mathbf{x}_{\text{LSLE}} &= \arg\min\{g(\mathbf{x}) \;:\; \mathbf{x} \in D_\alpha\} \\ &= \arg\min\{g(\mathbf{x}) \;:\; \text{depth}(\mathbf{x}) \geq 1 - \alpha\}\end{aligned}$$

In the example we set $g(\mathbf{x}) = 25 - 3x_1 - 5x_2$ (large values of exponential risk factors lead to ruin).

Two Exponential Risk Factors



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Convexity Theory and LSLEs

- The depth set D_α is a closed convex set (an intersection of closed half planes).
- We can define the support function

$$\phi(\mathbf{u}) = \sup\left\{\mathbf{u}'\mathbf{x} \;:\; \mathbf{x} \in D_\alpha\right\}$$

  and it is well known that this must be convex and positive homogeneous, hence sub-additive.
- In general we can show that

$$\phi(\mathbf{u}) \leq q_\alpha(\mathbf{u}'\mathbf{X})$$

  with equality if and only if $q_\alpha(\cdot)$ is a sub-additive risk measure on the space $\{\mathbf{u}'\mathbf{X} : \mathbf{u} \in \mathbb{R}^d\}$.
- Thus the LSLE gives a lower bound for the α-VaR of the loss (under linear impacts).

[Rockafellar, 1970]

More Theory

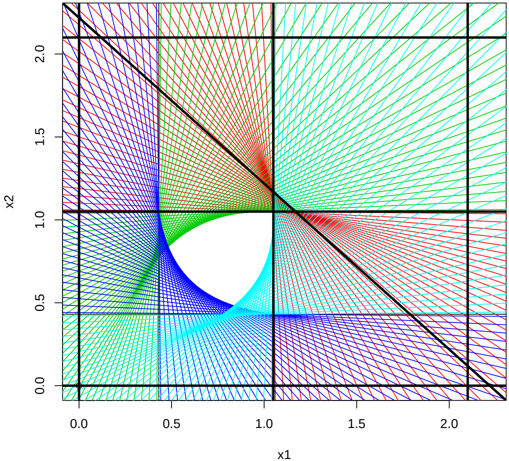
- We can substitute $-\mathbf{u}$ for $\mathbf{u}$ to conclude that

$$q_{1-\alpha}(\mathbf{u}'\mathbf{X}) \leq \inf\left\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in D_\alpha\right\} = -\phi(-\mathbf{u})\ .$$

- Thus the LSLE gives an upper bound for the $(1-\alpha)$-quantile of the net asset value distribution (under linear impacts).
- The situations leading to non-subadditivity can be diagnosed by lack of smoothness of the boundary of the depth set.
- Note that is also possible to find an outer scenario set O_α such that

$$\sup\left\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in D_\alpha\right\} \leq q_\alpha(\mathbf{u}'\mathbf{X}) \leq \sup\left\{\mathbf{u}'\mathbf{x} \,:\, \mathbf{x} \in O_\alpha\right\}\ ,$$

Two Independent Exponentials, $D_{0.65}$



Non-Subadditivity of Quantiles

- In previous slide we set $\alpha = 0.65$ and $u_1 = u_2 = 1$.
- Diagonal line is $x_1 + x_2 = q_\alpha(X_1 + X_2)$ which obviously intersects axes at $(0, q_\alpha(X_1 + X_2))$ and $(q_\alpha(X_1 + X_2), 0)$.
- Horizontal (vertical) lines are at 0, $q_\alpha(X_1)$ and $2q_\alpha(X_1)$.
- We infer
  1. $x_1 + x_2 < q_\alpha(X_1 + X_2)$ in the depth set;
  2. $\sup\{x_1 + x_2 \,:\, \mathbf{x} \in D_\alpha\}$ is a poor lower bound
  3. $q_\alpha(X_1 + X_2) > q_\alpha(X_1) + q_\alpha(X_2)$ (non-subadditivity of quantile risk measure)
  4. By setting $u_1 = u_2 = -1$ we could have concluded that $q_{1-\alpha}(X_1 + X_2) < q_{1-\alpha}(X_1) + q_{1-\alpha}(X_2)$

Concluding Remarks

1. When do depth sets, LSLES and MLREs have the most useful theoretical properties?
   - When we consider linear impacts of risk factors.
   - When VaR is subadditive on $\{\mathbf{u}'\mathbf{X} : \mathbf{u} \in \mathbb{R}^d\}$.
2. Is it easy to compute depth sets and LSLEs in practice?
   - Analytical solution for linear impacts and elliptical risk factors.
   - Otherwise depth sets and stress events have to be calculated numerically in general.
   - Very difficult to calculate depth sets for general multivariate distributions with mixed marginal distributions.

More Concluding Remarks

3. What about approximate application of the theory?
   - It makes sense to look for best approximating elliptical distribution to determine depth sets.
   - For non-linear impact functions, still require a numerical search to find LSLE/MLRE.
   - Can linearize impact function for added simplicity (but may miss important non-linear effects).
4. Can the method be used in a Monte Carlo or ESG context?
   - Can use simulated risk factor output to estimate best-fitting elliptical model.
   - Not feasible to calculate empirical depth set (bagplot) in higher dimensions.
   - If impact on net assets is simulated, can also estimate shape of impact function (or look for linear approximation using regression).

For Further Reading

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For Further Reading (cont.)

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