Dealing With Vine's Issues

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Overview

General Copula Definition

Definition: Copulas

- Let $X = [X_1, X_2, ..., X_n]$ be a vector of random variables:
  - of distribution function F,
  - of joint density f,
  - of marginal distribution functions $F_1, F_2, ..., F_n$,
  - of marginal densities $f_1, f_2, ..., f_n$.
- Sklar (1959)'s theorem insures the existence of two functions c and C such that:

$$\begin{aligned} F(x) &= C(F_1(x_1), F_2(x_2), ..., F_n(x_n)) \\ f(x) &= \prod_{i=1}^{n} f_i(x_i) \cdot c(F_1(x_1), F_2(x_2), ..., F_n(x_n)) \end{aligned}$$

Vine Copula Definition

Definition: Vine Copulas

A vine copula is a n-variate parametric copula built by decomposing the multivariate density in a product of bivariate copulas.

- Origins: Joe (1997) and Bedford and Cooke (2002, 2001)
- Applications: Aas et al. (2009), Berg and Aas (2009), Czado et al. (2009) and Guégan and Maugis (2009).

Decomposition Example In Dimension 3

$$
\begin{aligned}
f(x_1, x_2, x_3) &= f(x_1) \cdot f(x_2) \cdot f(x_3) \\
&\cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{13}(F_1(x_1), F_3(x_3)) \\
&\cdot c_{23|1}(\partial_1 C_{12}(F_1(x_1), F_2(x_2)), \partial_1 C_{13}(F_1(x_1), F_2(x_2)))
\end{aligned}
$$

Trivariate Vine Copula

$$
\begin{aligned}
c_{123} &= c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{13}(F_1(x_1), F_3(x_3)) \\
&\cdot c_{23|1}(\partial_1 C_{12}(F_1(x_1), F_2(x_2)), \partial_1 C_{13}(F_1(x_1), F_2(x_2))) \\
c_{123} &= c_{12} \cdot c_{13} \cdot c_{23|1}
\end{aligned}
$$

Selection Issue

Issues:

- Number of vine copulas:

$$\frac{n!}{2}\prod_{i=1}^{n-3} i!$$

- Copula Selection: Chen et al. (2004) and Chen and Fan (2006) and Kiefer and Choi (2006).

Lattice Selection:

Gabriel (1969), Andersson (1988), Edwards and Havranek (1987) and Antoch and Hanousek (2000)

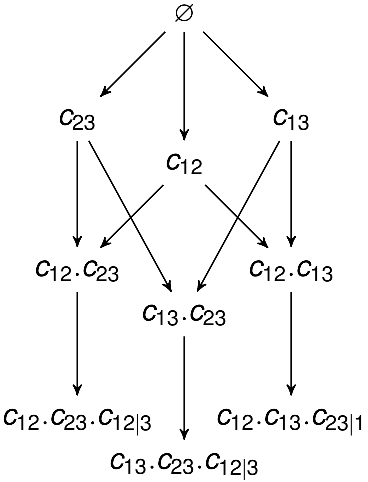
Specification & Generalization: Lattice structure

Among models, some can be considered as specifications of others, and some as generalizations of others.

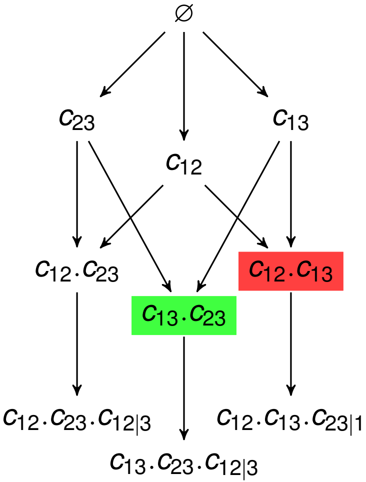
Coherence Principle:

Specifications of a false model are false, and generalizations of valid models are valid.

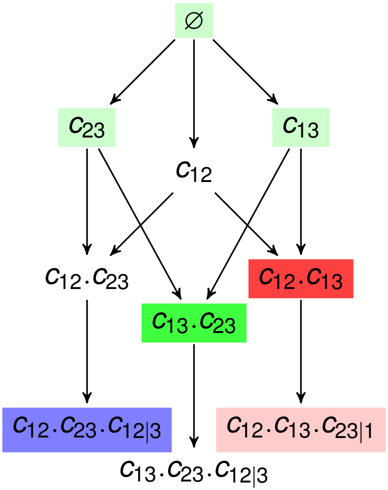
Example


∅
C_{23}
C_{12}
C_{13}
$C_{12}.C_{23}$
$C_{13}.C_{23}$
$C_{12}.C_{13}$
$C_{12}.C_{23}.C_{12|3}$
$C_{13}.C_{23}.C_{12|3}$
$C_{12}.C_{13}.C_{23|1}$

Example

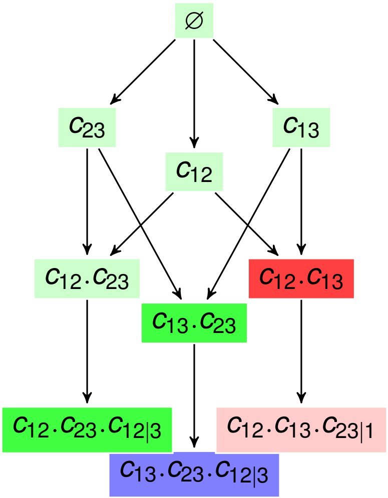

∅
c_{23}
c_{13}
c_{12}
$c_{12} \cdot c_{23}$
$c_{12} \cdot c_{13}$
$c_{13} \cdot c_{23}$
$c_{12} \cdot c_{23} \cdot c_{12|3}$
$c_{12} \cdot c_{13} \cdot c_{23|1}$
$c_{13} \cdot c_{23} \cdot c_{12|3}$

Example



∅
c_{23}
c_{12}
c_{13}
$c_{12} \cdot c_{23}$
$c_{13} \cdot c_{23}$
$c_{12} \cdot c_{13}$
$c_{12} \cdot c_{23} \cdot c_{12|3}$
$c_{13} \cdot c_{23} \cdot c_{12|3}$
$c_{12} \cdot c_{13} \cdot c_{23|1}$

Example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C_{23}
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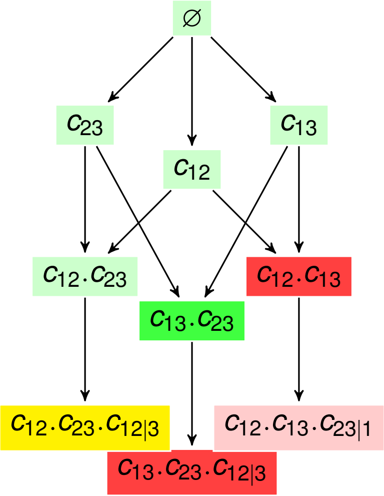
Example



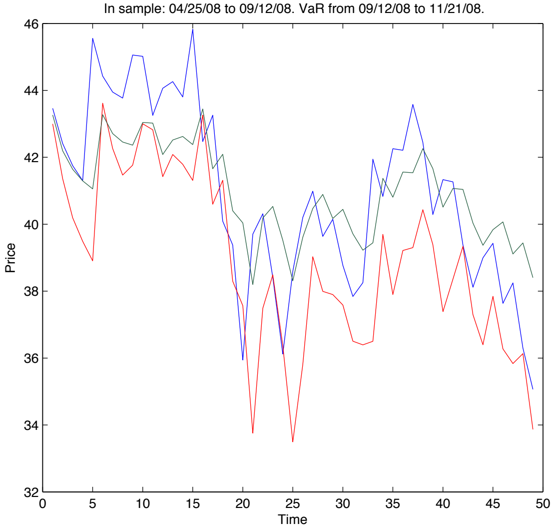
$\varnothing$
c_{23}
c_{12}
c_{13}
$c_{12}.c_{23}$
$c_{13}.c_{23}$
$c_{12}.c_{13}$
$c_{12}.c_{23}.c_{12|3}$
$c_{12}.c_{13}.c_{23|1}$
$c_{13}.c_{23}.c_{12|3}$

CAC40

Procedure:

- Five main assets composing the *CAC*40 from 25/04/08 to 21/11/08 (Datastream).
- In-sample: 04/25/08 to 09/12/08.
  Out-Sample: 09/12/08 to 11/21/08.
- Fully automated estimation and search.
  - Univariate process: *GARCH* using *AIC*.
  - Multivariate estimation: Maximum Likelihood.
  - Test: Modified Anderson Darling (Chen et al., 2004).

VaR



In sample: 04/25/08 to 09/12/08. VaR from 09/12/08 to 11/21/08.
Price
Time
46
44
42
40
38
36
34
32
0
5
10
15
20
25
30
35
40
45
50

Summary

- Vine copulas are parametric copulas built using bivariate copulas.
- Vine copulas can be selected using a lattice based search algorithm.
- By completely modeling the multivariate distribution, Vine copulas permits to build better risk measures.

Issues & Prospects

Issues:

Heavily Computational.

Prospects:

Better bivariate copulas.

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Annex 1

$$
\begin{aligned}
f(x_1, x_2, x_3) &= f(x_1, x_2) \cdot \frac{f(x_2, x_3 | x_1)}{f(x_2 | x_1)} \\
f(x_1, x_2, x_3) &= f(x_1) \cdot f(x_2) \cdot c_{12}(F_1(x_1), F_2(x_2)) \\
&\quad \cdot \frac{f(x_2|x_1) \cdot f(x_3|x_1)}{f(x_2|x_1)} \cdot c_{23|1}(F_{2|1}(x_2|x_1), F_{3|1}(x_3|x_1)) \\
f(x_1, x_2, x_3) &= f(x_1) \cdot f(x_2) \cdot c_{12}(F_1(x_1), F_2(x_2)) \\
&\quad \cdot \frac{f(x_1, x_3)}{f(x_1)} \cdot c_{23|1}(\partial_1 C_{12}(F_1(x_1), F_2(x_2)), \partial_1 C_{13}(F_1(x_1), F_2(x_2))) \\
f(x_1, x_2, x_3) &= f(x_1) \cdot f(x_2) \cdot f(x_3) \\
&\quad \cdot c_{12}(F_1(x_1), F_2(x_2)) \cdot c_{13}(F_1(x_1), F_3(x_3)) \\
&\quad \cdot c_{23|1}(\partial_1 C_{12}(F_1(x_1), F_2(x_2)), \partial_1 C_{13}(F_1(x_1), F_2(x_2)))
\end{aligned}
$$

Annex 2 I

The test is the following: we are testing the null hypothesis H_0 against the alternative hypothesis H_1:

$$H_0 : Pr(C(U_1, \dots, U_n) = C_0(U_1, \dots, U_n)) = 1$$

$$H_1 : Pr(C(U_1, \dots, U_n) = C_0(U_1, \dots, U_n)) < 1$$

where C is the true copula and C_0 the estimated copula. We define then $\{Z_i\}_{i<n}$ and W as:

$$Z_i = F(U_i | U_1, \dots, U_{i-1}) \text{ and } W = \sum_1^n [\Phi^{-1}(Z_j)]^2,$$

where Φ is the standard normal distribution function. Then W follows a χ^2_d distribution; our test will be based on this result. For this purpose we will use this univariate boundary kernel $K_h(x, y)$

$$K_h(x, y) = \begin{cases} k(\frac{x-y}{h}) / \int_{-\frac{x}{h}}^{1} k(u)du & \text{if } x \in [0, h) \\ k(\frac{x-y}{h}) & \text{if } x \in [h, 1-h] \\ k(\frac{x-y}{h}) / \int_{-\frac{x}{h}}^{1} k(u)du & \text{if } x \in (1-h, 1] \end{cases}$$

Annex 2 II

in which $k(\cdot)$ is the quartic kernel: $k(u) = \frac{15}{16}(1-u^2)^2\mathbf{1}_{|u|<1}$. Then we define g_W as a kernel estimation of its density function, where h is fixed using the rule of thumb ($h = \sqrt{Var(W)}n^{-1/5}$):

$$g_W(\omega) = \frac{1}{n * h}\sum_t^T K_h(\omega, F_{\chi^2_d}(W_t))$$

The alternative test is based on:

$$J_n = \int_0^1 [g_W(\omega) - 1]^2 \, d\omega$$

Then under classical hypothesis we have that:

$$Stat_n = \frac{(nh^{1/2}J_n - r_n)}{\sigma} \rightarrow N(0,1) \text{ in distribution.}$$

Annex 2 III

Where:

$$r_n = h1/2\left[(h^{-1} - 2)\int_{-1}^{1} k^2(\omega)d\omega + 2\int_{0}^{1}\int_{-1}^{z} k_z^2(y)dydz\right]$$

$$\sigma^2 = 2\int_{-1}^{1}\left[\int_{-1}^{1} k(u+v)k(v)dv\right]^2 du$$

$$k_z(y) = k(y)/\int_{-1}^{z} k(u)du$$

For further details on the test, including Monte-Carlo estimation of the power of the test, see Chen et al. (2004).