Modeling Contagious Credit Events and Risk Analysis of CDO

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Model framework

Modeling Credit Events and
its Application to Risk Analysis of Credit Portfolio

- Credit events such as Rating changes and Defaults
- Based on top-down approach
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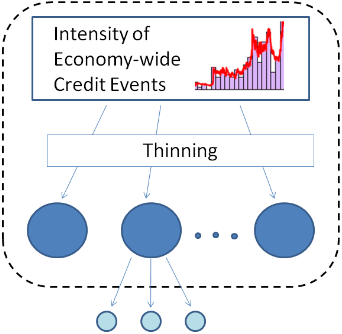


Related previous works

- Modeling defaults with top-down approach
  - Giesecke et al. (2005)
    - Mathematical aspects of top-down approach
  - Giesecke and Kim (2009)
    - Self-exciting intensity with state-dependent property
    - Risk analysis of CDO
- Modeling rating changes with top-down approach
  - Nakagawa (2008)
    - Mutually exciting intensity process
    - Multi-downgrade protection

Our model is based on Nakagawa (2008) and Giesecke and Kim (2009).

Intensity process

- $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\})$: a filtered complete probability space
- $0 < T_1^l < T_2^l < \cdots$: $\{\mathcal{F}_t\}$-adapted point process
  - T_n^l: the n-th event time of event type l, totally inaccessible stopping time.
  - the event $l = 1, 2$ indicates up-grade and down grade respectively.
  - the event $l = 3$ indicates default.
- $N_t^l = \sum_{n \geq 1} \mathbf{1}_{\{T_n^l \leq t\}}$: the counting process of event l.

- λ_t^l: the intensity process associated with N_t^l for each l.
  - λ_t^l is an $\{\mathcal{F}_t\}$-progressively measurable, nonnegative process such that $N_t^l - \int_0^t \lambda_s^l \mathrm{d}s$ is an $\{\mathcal{F}_t\}$-martingale.

Ideas for modeling credit events intensity

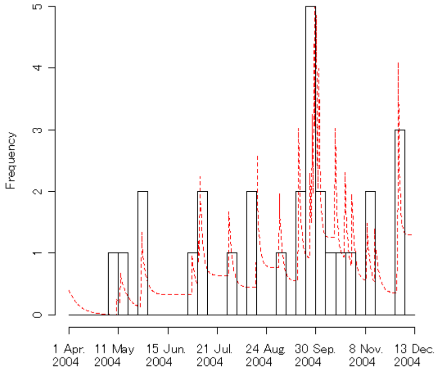


Fig.: The number of up-grades and estimated up-grade intensity path. The bar chart displays the number of up-grades in 2004, in Japan, and the broken line graph is the estimated path of the up-grade intensity.

Features of credit events:

Clusters of credit events are observed.

To capture the feature,

we consider intensity with self-exciting property.

Intensity is in high level with event clusters [a].

[a] Nakagawa (2009) finds self-exciting effects in rating changes of Japanese firms during April 1998 to September 2009 with mutually exciting intensity model, which is an extension of the self-exciting models.

Self-exciting intensity process

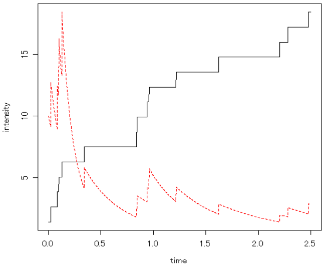
Self-exciting intensity model with state-dependent property

$$d\lambda_t^l = \kappa_t^l(c_t^l - \lambda_t^l)dt + dJ_t^l.$$

Self-exciting property:

Intensity jumps when the event occurs

$$J_t^l = \sum_{n\geq 1}(\min(\delta^l \lambda_{T_n^l -}^l,\ \gamma^l)\mathbf{1}_{\{T_n^l \leq t\}})$$



State-dependent property:

Reversion speed: $\kappa_t^l = \kappa^l \lambda_{T^l_{N_t^l}}^l$

Reversion level: $c_t^l = c^l \lambda_{T^l_{N_t^l}}^l$

Difference from Giesecke and Kim (2009):

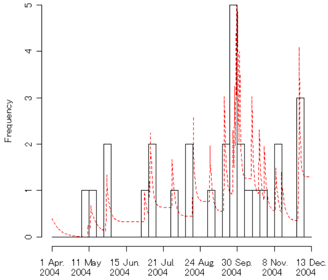
- Three types events (up-grade, down-grade and default).
- Jump size of intensity is bounded.

Tentative estimation result for intensity model

Table: Estimated parameters of the intensity processes

	κ	c	γ	λ_0	p-value
up-grade ($l = 1$)	1.745	0.350	90.804	26.486	0.062
down-grade ($l = 2$)	1.643	0.281	168.839	82.676	0.063
default ($l = 3$)	3.450	0.503	23.384	1.181	0.974

We fixed the parameter δ as $\delta = 1.2$.



We apply the maximum likelihood method executed in Giesecke and Kim(2009). The data is rating change data of Japanese companies between 2004/4 and 2009/4. The ratings are announced by R& I. Since there are no defaults in the data, we treat the rating below BBB- as quasi-default.

We execute K-S test, and obtain p-values.

P-values indicate that the models are not rejected at 5% significant level.

Random thinning

Intensities of up-grade and down-grade from k to k' in portfolio S_i:

$$\lambda_t^{(i,1)}(k,\ k') = Z_t^{(i,1)}(k,\ k')\lambda_t^1,$$
$$\lambda_t^{(i,2)}(k,\ k') = Z_t^{(i,2)}(k,\ k')\lambda_t^2.$$

$k = 1, 2, \cdots, K$: credit ratings.

$Z_t^{i,l}(k,\ k')$: conditional probability that the rating change from the rating k to the rating k' of the firm in the portfolio S_i, if a rating change occurs in the economy.

$$Z_t^{(i,l)}(k,\ k') := \lim_{\varepsilon\to 0} Z_t^{(i,l)}(k,\ k',\varepsilon),$$
$$Z_t^{(i,l)}(k,\ k',\varepsilon) = \sum_n \frac{P[\{T_n^l \in \tau^l(S_i)\}\cap\{T_n^l \in \tau^l(k,\ k')\}\cap\{T_n^l \le t+\varepsilon\}|\mathcal{F}_t]}{P[T_n^l \le t+\varepsilon|\mathcal{F}_t]}\mathbf{1}_{\{T_{n-1}^l < t \le T_n^l\}}.$$

Intensity of defaults from k in portfolio S_i :

$$\lambda_t^{(i,3)}(k) = Z_t^{(i,3)}(k)\lambda_t^3.$$

$Z_t^{(i,3)}(k)$: conditional probability that the defaulter is the k-rated firm in the portfolio S_i, if default occurs in the economy.

Model of thinning

We specify the thinning with rating distribution:

$$
\begin{aligned}
Z_t^{(i,1)}(k,\ k') &= \frac{X_t^{(i)}(k)}{\sum_{\tilde{k}=(k-k')+1}^{K} X_t^*(\tilde{k})} z_{k-k'}^{1} \mathbf{1}_{\{\sum_{\tilde{k}=(k-k')+1}^{K} X_t^*(\tilde{k})>0\}} \\
Z_t^{(i,2)}(k,\ k') &= \frac{X_t^{(i)}(k)}{\sum_{\tilde{k}=1}^{K-(k'-k)} X_t^*(\tilde{k})} z_{k'-k}^{2} \mathbf{1}_{\{\sum_{\tilde{k}=1}^{K-(k'-k)} X_t^*(\tilde{k})>0\}} \\
Z_t^{(i,3)}(k) &= \frac{X_t^{(i)}(k)}{X_t^*(k)} z_k^3 \mathbf{1}_{\{X_t^*(k)>0\}}
\end{aligned}
$$

- $X_t^{(i)}(k)$: the number at time t of k-rated firms in the portfolio S_i.
- $X_t^*(k)$: the number at time t of k-rated firms in the whole economy.
- $z_m^l(l = 1, 2)$: conditional probability that the rating change is the m-step rating change, if the rating change occurs.
- z_k^3: the conditional probability that the rating of a defaulter is k, if the default occurs

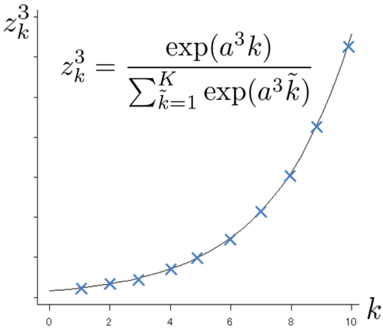
Parameters of thinning model

- $z_m^l (l = 1, 2)$: conditional probability that the rating change is the m-step rating change, if the rating change occurs.
  - $z_1^l > z_2^l > \ldots > z_{K-1}^l \ (l = 1,\ 2)$.
- z_k^3 : the conditional probability that the rating of a defaulter is k, if a default occurs
  - $0 < z_1^3 < z_2^3 < \ldots < z_K^3 < 1$.

We model z_m^1, z_m^2, z_k^3 as follows :

$$z_m^l = \frac{\exp(a^l(K-m))}{\sum_{\tilde{m}} \exp(a^l(K-\tilde{m}))} \quad (l{=}1,2),$$

$$z_k^3 = \frac{\exp(a^3 k)}{\sum_{\tilde{k}} \exp(a^3 \tilde{k})}.$$



Estimated parameters of thinning model (rating change)

- We obtain $\hat{z}^l_m = \exp(a^l(K-m))/\sum_{\tilde{m}} \exp(a^l(K-\tilde{m}))$ by minimizing $\sum_m (\tilde{z}^l_m - \hat{z}^l_m)^2 \quad (l=1,2)$.
  - $\tilde{z}^1_m$: Ratio of m-step up-grades to all up-grades obtained from data.
  - $\tilde{z}^2_m$: Ratio of m-step down-grades to all down-grades obtained from data.

m	$\tilde{z}^1_m$	$\tilde{z}^2_m$
1	0.9024	0.8618
2	0.0732	0.1290
3	0.0244	0.0046
4	0	0.0046
5	0	0
6	0	0
7	0	0
8	0	0
9	0	0

m	$\hat{z}^1_m$	$\hat{z}^2_m$
1	0.9024	0.8618
2	0.08808	0.1191
3	0.008597	0.01646
4	8.400×10^{-4}	0.002276
5	8.289×10^{-5}	3.154×10^{-4}
6	8.992×10^{-6}	4.444×10^{-5}
7	1.780×10^{-6}	6.999×10^{-6}
8	1.076×10^{-6}	1.824×10^{-6}
9	1.007×10^{-6}	1.109×10^{-6}

We used the data from April 1, 1998 to April 1, 2009 on rating changes. Ratings are $k = 1, 2, \cdots, 10$.

$a^1 = 2.327, a^2 = 1.979$

Estimated parameters of thinning model (default)

- We obtain $\hat{z}^3_k = \exp(a^3 k)/\sum_{\tilde{k}} \exp(a^3 \tilde{k})$ by minimizing $\sum_k (\tilde{z}^3_k - \hat{z}^3_k)^2$.
  $\tilde{z}^3_k$: Ratio of defaults from rating k to all defaults obtained from data.

k	$\tilde{z}^3_k$
1	0
2	0
3	0
4	0
5	0
6	0
7	0
8	0.03
9	0.26
10	0.71

k	$\hat{z}^3_k$
1	1.009×10^{-5}
2	3.531×10^{-5}
3	1.223×10^{-4}
4	4.222×10^{-4}
5	0.001456
6	0.005022
7	0.01732
8	0.05972
9	0.2059
10	0.7100

$a^3 = 1.238$

We used the data from April 1, 1998 to April 1, 2009 on rating changes.

Outline of simulation algorithm

$$^{\forall}t \in [0, T] \qquad \bar{\lambda}_t = \sum_l \lambda_t^l \leq \tilde{\lambda}$$

Generate candidate event times with intensity $\tilde{\lambda}$ (const.)

$$0 < \tilde{T}_1 < \tilde{T}_2 < \tilde{T}_3 < \tilde{T}_4 < \tilde{T}_5 \cdots$$

Accept with probability $\bar{\lambda}_t / \tilde{\lambda}$ (Ogata1981)

$$T_1 \qquad T_2 \quad \cdots$$

Specify the event with probability $\lambda_t^l / \bar{\lambda}_t \times Z_t^{(i,l)}$

- up-grade
- from rating 3 to rating 2
- portfolio1

- default
- from rating 8
- portfolio3

Setting of numerical example

We derive loss distributions of loan portfolios from simulation.

- Each portfolio consists of loans of 100 firms.
  - Portfolio 1 : High credit quality
  - Portfolio 2 : Averaged credit quality
  - Portfolio 3 : Low credit quality
- Maturity, loan amount and recovery rate are common (recovery rate = 40 %).

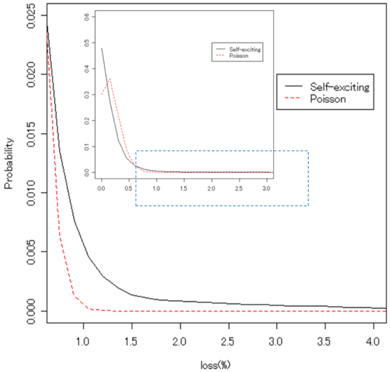
Table: Rating distribution of portfolios

rating	1	2	3	4	5	6	7	8	9	10
portfolio 1	15	15	15	15	15	5	5	5	5	5
portfolio 2	10	10	10	10	10	10	10	10	10	10
portfolio 3	5	5	5	5	5	15	15	15	15	15
residual	10	10	10	10	10	10	10	10	10	10

We used estimated parameter values.

Economy-wide loss distribution

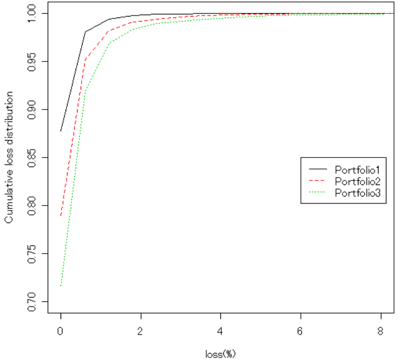
One year loss distribution obtained by self-exciting intensity (our model) and that of constant intensity (Poisson model).



	Self-exciting	Poisson
Average	0.18%	0.18%
Max	8.25%	1.50%
99.9%VaR	4.35%	0.90%

Self-exciting property $\longrightarrow$ Fat-tail

Loss distribution of portfolio



	Port.1	Port.2	Port.3
99%VaR	1.20%	1.80%	3.00%
99%ES	1.79%	3.25%	4.66%

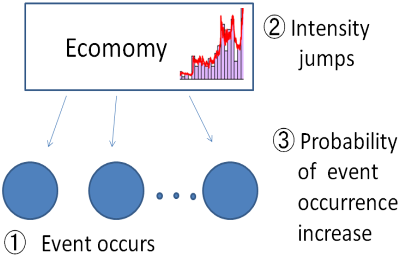
One year loss of portfolio 1,2 and 3.

As credit quality of portfolio become lower,
the loss of portfolio increases.
← Thinning model

The condition $z_1^3 < z_2^3 < \cdots < z_{10}^3$ makes the default intensity of portfolio 3 larger than that of others.

Risk contagion in the model (1)

Feature of Model : Capture risk contagion among portfolios.



One year loss		
Port.2	Port.1	Port.3
0%-0.75%	0.07%	0.20%
0.75%-1.5%	0.24%	0.72%
1.5%-	0.92%	2.76%

As loss of portfolio 2 increases,

E [loss of port.1 | loss of port.2]
and
E [loss of port.3 | loss of port.2] .

also increases.

Risk contagion in the model (2)

Risk contagion from the first one year loss of portfolio 2
to the second one year loss of other portfolios

Table: Self-exciting model

First one year loss of port.2	Second one year loss			
	Economy	Port. 1	Port. 2	Port. 3
0%-0.75%	0.26%	0.13%	0.27%	0.39%
0.75%-1.5%	1.56%	0.78%	1.55%	2.37%
1.5%-	3.72%	1.85%	3.56%	5.74%

Table: Poisson model (constant intensity)

Loss of Port.2	Economy	Port.1	Port.2	Port.3
0%-0.75%	0.18%	0.09%	0.18%	0.28%
0.75%-1.5%				
1.5%-				

Setting of CDO

We consider three CBOs (CBO No.1, No.2 and No.3) with different reference portfolios.

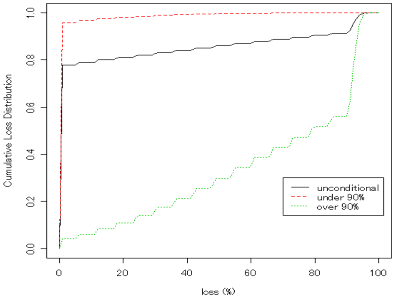
- Reference portfolio
  - Consist of 300 fixed rate bonds (Maturity: 5 years. Coupon rate: 4%).
- CBO
  - Tranching: Equity $[0\%, 5\%]$, junior mezzanine $[5\%, 10\%]$, senior mezzanine $[10\%, 15\%]$, senior $[15\%, 100\%]$
  - Coupon: junior mezzanine 2.5%, senior mezzanine 2.0%, senior 1.0%
  - Equity receives the residual of all coupons from the reference portfolio after superior tranches have received their coupons.
- CDO-squared with reference portfolio consisted by junior mezzanine of CBOs.

Table: Rating distributions of the reference portfolios

rating	1	2	3	4	5	6	7	8	9	10
portfolio 1	50	50	50	50	50	10	10	10	10	10
portfolio 2	30	30	30	30	30	30	30	30	30	30
portfolio 3	10	10	10	10	10	50	50	50	50	50
residual	10	10	10	10	10	10	10	10	10	10

Risk Contagion between different CDOs Tranches

Risk contagion between CBOs



- Conditional loss distributions of junior mezzanine of CBO No.2 on the conditions that loss of junior mezzanine of CBO No.3 is
  - under 90%
  - over 90%.

As loss of CBO No.3 increases,

$$\mathrm{E}\left[\text{loss of CBO No.2} \mid \text{loss of CBO No.3}\right]$$

also increase.

Sensitivity analysis of CDO-squared

Sensitivity of the risk measures of tranche loss of the CDO-squared to κ^3

κ^3: the decline speed parameter of the default intensity λ_t^3.

Table: Parameter sensitivity of Risk measures of the Junior Mezzanine of CDO-squared

κ^3	Average loss	99% VaR	99% ES	Maximum loss
2.45 (−1.00)	28.71% (+11.10%)	95.74% (+0.81%)	96.26% (+0.79%)	98.26% (+0.37%)
3.45 (±0.00)	17.60% (±0.00%)	94.93% (±0.00%)	95.47% (±0.00%)	97.89% (±0.00%)
4.45 (+1.00)	11.40% (−6.20%)	93.93% (−1.00%)	94.77% (−0.70%)	97.56% (−0.33%)

As κ^3 decreases, the default risk increases.

Concluding remarks

- Model of credit events based on the top-down approach.
  - Self-exciting intensities for Economy-wide credit event
    - state-dependent property
    - limited jump size
  - Random thinning processes
    - Distribution of credit ratings in the sub-portfolios
- Feature:
  - Possible to evaluate several portfolios simultaneously
  - Capture credit risk contagion among several portfolios
- Application
  - Tractable for risk analysis of credit portfolios

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