Pricing basket default swaps in a tractable shot-noise model

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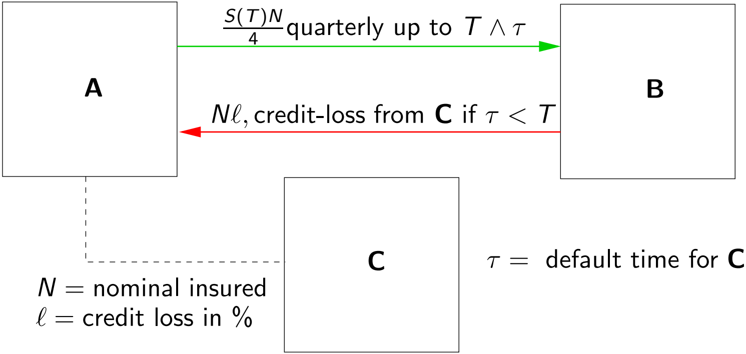
Content

- Give a short introduction to k-th-to default swaps
- Describe the quantities needed to price k-th-to default swaps in portfolios with homogeneous recoveries.
- Present a computational tractable shot-noise model in a homogeneous portfolio.
- Provide some details regarding the quantities needed for pricing k-th-to default swaps in the shot-noise model
- Present some numerical examples for k-th-to default swaps in our calibrated model

The Credit Default Swap (CDS)

- **C** have issued bonds. **C** defaults at time τ.
- **A** buys protection from **B**, on credit-losses inferred by **C**, for the next T years.



A

$\frac{S(T)N}{4}$quarterly up to $T \wedge \tau$

B

$N\ell$, credit-loss from **C** if $\tau < T$

C

$\tau =$ default time for **C**

$N =$ nominal insured
$\ell =$ credit loss in %

CDS, credit default swap cont.

$S(T)$ set so expected discounted cash-flows between **A** are **B** equal at $t = 0$,

$$S(T) = \frac{\mathbb{E}\left[1_{\{\tau \leq T\}} D(\tau)(1-\phi)\right]}{\sum_{n=1}^{4T} \mathbb{E}\left[D(t_n)\frac{1}{4}1_{\{\tau > t_n\}}\right]}$$

where $t_n = \frac{n}{4}$, $D(t) = \exp\left(-\int_0^t r_s ds\right)$ and r_t is risk-free interest rate at time t and $\ell = 1 - \phi$. The accrued premium at default is ignored.

If τ and r_t independent, and $1 - \phi$ deterministic, then

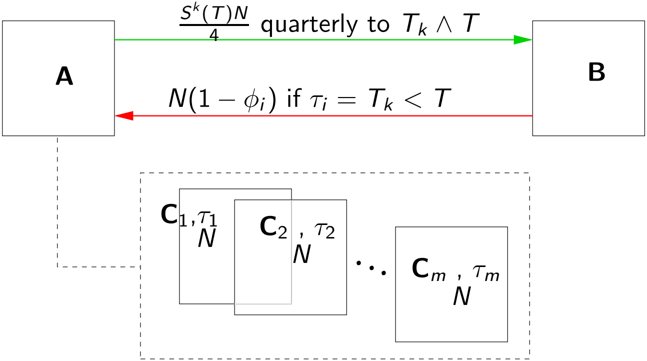
$$S(T) = \frac{(1-\phi)\int_0^T B_s dF(s)}{\frac{1}{4}\sum_{n=1}^{4T} B_{t_n}(1 - F(t_n))}$$

where $B_t = \mathbb{E}\left[D(t)\right]$ and $F(t) = \mathbb{Q}\left[\tau \leq t\right]$. Note that $S(T)$ don't depend on N.

The CDS is the most liquid single-name credit derivative and is also a main calibration tool in most credit derivatives models.

Portfolio credit derivatives: The k^{th}-to-default swap

- The k^{th}-to-default swap is a generalization of the CDS.
- The obligors $\mathbf{C}_1, \ldots, \mathbf{C}_m$ have issued bonds with default times $\tau_1, \tau_2, \ldots, \tau_m$.
- **A** buys protection from **B**, against k-th default among $\mathbf{C}_1, \ldots, \mathbf{C}_m$, up to T.
- The k-th default occurs at T_k where $\{T_n\}$ is the ordering of $\{\tau_i\}$.



The k^{th}-to-default spread $S^k(T)$ needs dependence model

$S^k(T)$ set so expected discounted cash-flows between **A** are **B** equal at $t = 0$,

$$S^k(T) = \frac{\sum_{i=1}^{m} \mathbb{E}\left[1_{\{T_k \le T\}} D(T_k)(1-\phi_i) 1_{\{T_k = \tau_i\}}\right]}{\sum_{n=1}^{n_T} \mathbb{E}\left[D(t_n)\frac{1}{4} 1_{\{T_k > t_n\}}\right]},$$

where $t_n = \frac{n}{4}$, $D(t) = \exp\left(-\int_0^t r_s ds\right)$ and r_t is risk-free interest rate at time t. The accrued premium at default is ignored.

If $\{\tau_i\}$ and r_t independent, and all recoveries are the same and given by the constant ϕ, that is $\phi_1 = \phi_2 = \ldots = \phi_m = \phi$, then

$$S^k(T) = \frac{(1-\phi)\int_0^T B_s dF_k(s)}{\frac{1}{4}\sum_{n=1}^{n_T} B_{t_n}(1 - F_k(t_n))}.$$

where $B_t = \mathbb{E}\left[D(t)\right]$ and $F_k(t) = \mathbb{Q}\left[T_k \le t\right]$.

To find $\mathbb{Q}\left[T_k \le t\right]$ we need model for the joint distribution of $\tau_1, \tau_2, \ldots, \tau_m$.

Finding the ordered default distribution $\mathbb{Q}\left[T_k \leq t\right]$

- Next, define the counting process N_t as

$$N_t = \sum_{k=1}^{m} 1_{\{T_k \leq t\}}. \tag{1}$$

- Note that N_t counts the number of defaults that have occurred among the obligors $i = 1, 2, \ldots, m$ up to time t, that is $\{T_k \leq t\} = \{N_t \geq k\}$, and thus

$$\mathbb{Q}\left[T_k \leq t\right] = 1 - \sum_{j=0}^{k-1} \mathbb{Q}\left[N_t = j\right]. \tag{2}$$

- Hence, under homogeneous recoveries, $\{\mathbb{Q}\left[N_t = j\right]\}_{j=0}^{m}$ fully determines the distribution $\mathbb{Q}\left[T_k \leq t\right]$ and thus the k^{th}-to-default spread $S^k(T)$.
- So it is enough to study N_t, that is $\{\mathbb{Q}\left[N_t = j\right]\}_{j=0}^{m}$ for $t > 0$.

The model

- Consider a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{Q})$ where $\mathbb{Q}$ is a martingale. Below, all computations are under $\mathbb{Q}$.
- Let $\{X_{i,j}, Y_j : 1 \leq i \leq m, j \geq 1\}$ be independent nonnegative random variables where $X_{i,j}$ has distribution F_i and Y_j has distribution F_Y.
- Furthermore, M_t is a Poisson process with intensity ρ.
- Let $\{\lambda_{t,i}\}$ be m processes where each $\lambda_{t,i}$ satisfies the SDE

$$\begin{aligned} d\lambda_{t,i} &= -\delta_i \lambda_{t,i} dt + dC_{t,i} \\ C_{t,i} &= \sum_{j=1}^{M_t} Y_j X_{i,j} \end{aligned} \tag{3}$$

- At each jump time S_j of M, $\lambda_{t,i}$ jumps by the amount $Y_j X_{i,j}$, otherwise $\lambda_{t,i}$ decays exponentially with rate δ_i.
- This process is a Markovian **shot-noise** process
- The dependence structure of the multivariate process $(\lambda_{t,1}, \lambda_{t,2}, \ldots, \lambda_{t,m})$ is completely determined by M_t and the random variables $\{Y_j : j \geq 1\}$.

The model, cont.

- Consider a portfolio consisting of m obligors. The default time of obligor i is denoted by τ_i.
- Let $\mathcal{F}_t = \sigma(M_s : 0 \leq s \leq t, \{X_{i,j}, Y_j : 1 \leq i \leq m, 1 \leq j \leq M_t\})$.
- Let $E_1, \ldots, E_m$ be independent random variables, exponentially distributed with parameter one, and independent of $\mathcal{F}_\infty$. We define the default time τ_i as

  $$\tau_i = \inf\left\{ t > 0 : \int_0^t \lambda_{s,i} ds \geq E_i \right\},$$

  which implies that τ_i have default intensities λ_i w.r.t the filtration $\mathcal{F}_t$. Furthermore, for $T > t$, it is easy to see that

  $$\mathbb{Q}\left[\tau_i > t \,|\, \mathcal{F}_T\right] = \exp\left(-\int_0^t \lambda_{s,i} ds \right). \tag{4}$$

- Note that conditional on $\mathcal{F}_\infty$ the default times are independent.

The model, cont.

The following lemma will be useful and introduces notation needed later on.

Useful lemma: Lemma 1

Let $H_i(x) = \delta_i^{-1}(1 - e^{-\delta_i x})$. Then

$$\int_0^t \lambda_{s,i} ds = \lambda_{0,i} H_i(t) + \sum_{j=1}^{M_t} Y_j X_{i,j} H_i(t - S_j). \tag{5}$$

- Lemma 1 purely results from the shot-noise assumptions and can easily be generalized to non-exponential decay.
- It also holds for arbitrary random variables η_j replacing $Y_j X_{i,j}$.
- Lemma 1 holds both for inhomogeneous and homogeneous credit portfolios.

The model, cont.

- As seen earlier, to price portfolio k^{th}-to-default spreads on portfolios with homogeneous recoveries, it is enough to find the distribution $\{\mathbb{Q}[N_t = k]\}_{k=0}^{m}$ at different time points t,
- Furthermore, to price CDS spreads we need the individual default distributions $\mathbb{Q}[\tau_i \leq t]$.
- In this talk, we only consider exchangeable portfolios in our model given by (3). We therefore make the following assumption in (3)

$$\delta_i = \delta \quad \text{and} \quad F_i = F \quad \text{for} \quad 1 \leq i \leq m. \tag{6}$$

This implies that the default times $\{\tau_i\}$ are exchangeable and have the same distribution, and the exchangeability also implies that

$$\mathbb{Q}[N_t = k] = \binom{m}{k} \mathbb{Q}\left[\bigcap_{i=1}^{k} \{\tau_i \leq t\}, \bigcap_{i=k+1}^{m} \{\tau_i > t\}\right] \tag{7}$$

which reduces the computations to find $\mathbb{Q}\left[\bigcap_{i=1}^{k} \{\tau_i \leq t\}, \bigcap_{i=k+1}^{m} \{\tau_i > t\}\right]$.

The model, cont.

Due to the exchangeability, we can then state the following lemma, which is independent of the dynamics of the model (i.e. the default intensity process), as long as the model is exchangeable (and has the conditionally independence property)

Lemma 2:

Under (6) we have that

$$\mathbb{Q}\left[N_t = k\right] = \binom{m}{k} \sum_{j=0}^{k} \binom{k}{j} (-1)^j G(t, m-k+j). \tag{8}$$

where $G(t,k)$ denotes

$$G(t,k) = \mathbb{E}\left[e^{-\sum_{i=1}^{k} \int_0^t \lambda_{s,i} ds}\right]. \tag{9}$$

The proof is based on expanding $\mathbb{Q}\left[\bigcap_{i=1}^{k} \{\tau_i \leq t\}, \bigcap_{i=k+1}^{m} \{\tau_i > t\}\right]$.

The model, cont.

- Thus, in order to find $\mathbb{Q}\left[N_t = k\right]$ it is sufficient to compute $\{G(t, m-k+j)\}_{j=0}^{k}$ for any $k = 1, \ldots, m$.
- Below we denote by X a prototype for $X_{i,j}$ and similarly Y for Y_j.
- Recall that $\varphi_X(z) = \mathbb{E}\left[e^{-zX}\right]$ is the Laplace transform of the non-negative random variable X. Also recall that $H(x) = \delta^{-1}(1 - e^{-\delta x})$.
- Finally, note that $\lambda_{0,i} = \lambda_0$ for all obligors i, due to exchangeability.

The following theorem gives the necessary quantities for finding $\{G(t, j)\}_{j=1}^{m}$.

Theorem 1:

Under (6) we have that

$$G(t,k) = e^{-k\lambda_0 H(t) - \rho t} \cdot \exp\left(\rho t \int_{\mathbf{R}} \int_0^1 \left(\varphi_X\left(yH(tz)\right)\right)^k dz F_Y(dy)\right) \tag{10}$$

An explicit example

Assume that

$$Y \in \{y_1, y_2\} \quad \text{where} \quad \mathbb{Q}\left[Y = y_1\right] = q \quad \text{and} \quad X \sim \chi^2(2) \tag{11}$$

where $y_1, y_2 \geq 0$. Hence, Y is a two-point distributed random variable and X has chi-squared distribution with 2 degrees of freedom. Then

Lemma 3: An explicit example

Under (6) and (11) we have that

$$G(t,k) = \exp\Big(- k\lambda_0 H(t) + \rho t \left[q I(y_1,k,t) + (1-q) I(y_2,k,t) - 1 \right] \Big) \tag{12}$$

where $I(y,k,t)$ denotes

$$I(y,k,t) := \int_0^1 \frac{1}{\left(1 + 2y\delta^{-1}\left(1 - e^{-\delta t z}\right)\right)^k} dz. \tag{13}$$

The marginal survival distribution

For $k > 2$ it seems tedious to simplify $G(t,k)$ further. However, for $k = 1$, we can simplify (12) as stated in the following lemma.

Lemma 4: An explicit example of the marginal survival distribution

Under (6) and (11) we have that

$$\mathbb{Q}\left[\tau_i > t\right] = e^{-\lambda_0 H(t) + ct}\left[1 + 2y_1\delta^{-1}(1 - e^{-\delta t})\right]^{\frac{\rho q}{\delta + 2y_1}}\left[1 + 2y_2\delta^{-1}(1 - e^{-\delta t})\right]^{\frac{\rho(1-q)}{\delta + 2y_2}} \tag{14}$$

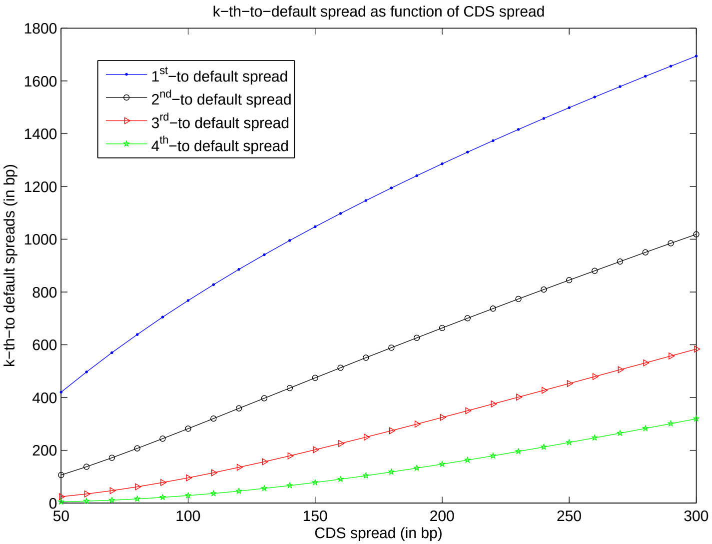
where c is given by

$$c = \rho\left(\frac{q}{1 + 2y_1\delta^{-1}} + \frac{1-q}{1 + 2y_2\delta^{-1}} - 1\right). \tag{15}$$

Recall that τ_i is the individual default time for obligor i, which by exchangeability has the same distribution for all obligors in the portfolio.

Note that $\mathbb{Q}\left[\tau_i > t\right]$ do not depend on the portfolio size m.

$S^k(5)$ as function of CDS-spread (basket of 10 obligors)



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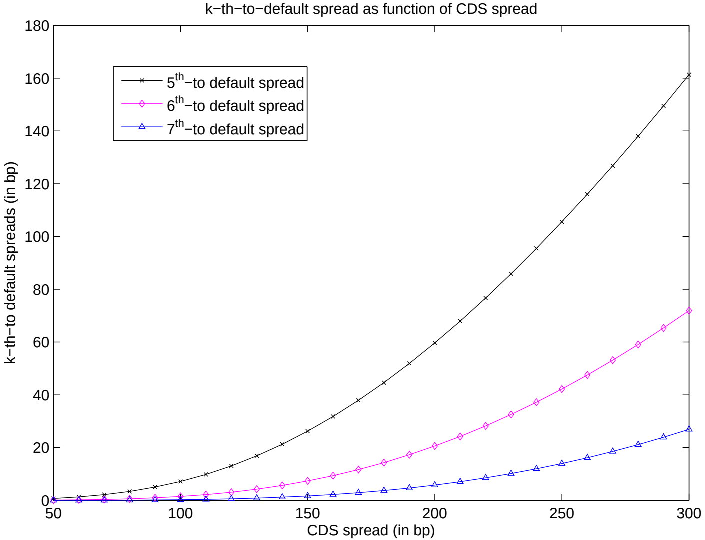
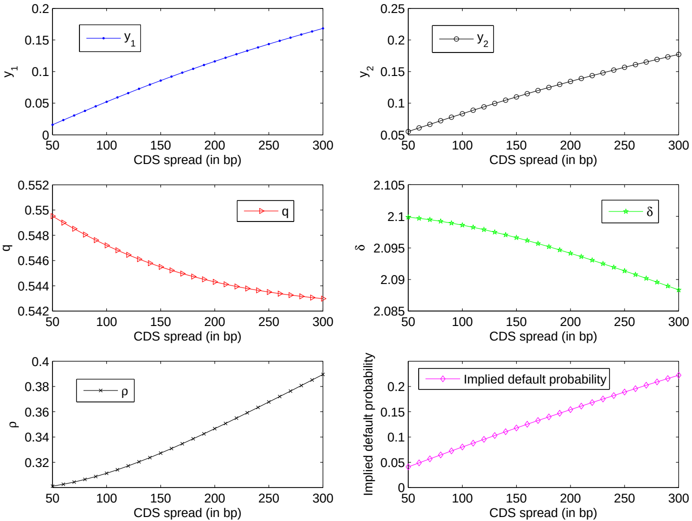


The calibrated parameters as function of CDS-spread



Thank you for your attention!